

Title:

To Infinity and Beyond



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0 Introduction

IMAGINE,
CREATE,
EXPLORE!!!

1 Infinite Arithmetic

$$\infty + 1 = \infty$$

$$\infty + k = \infty \text{ where } k \text{ is any real number}$$

$$\infty + \infty = \infty$$

$$\infty \cdot \infty = \infty$$

$$\infty \cdot k = \infty$$

$$\frac{1}{\infty} = 0$$

But

$$\frac{1}{\infty} = 0 \Rightarrow 0 \cdot \infty = 1 \text{ ?????}$$

OR

$$\frac{1}{\infty} = 0 \Rightarrow \infty = \frac{1}{0} \text{ ?????}$$

Question: What if there is also an infinitesimally small *number* ε such that $\varepsilon > 0$ and $\varepsilon^2 = 0$. Before you say “No way!!!!” think how ludicrous originally was the idea of an existence of a number i ;which has a property $i^2 = -1$.

Examples: How to use infinite arithmetic?

What happens with expression $\frac{1+n}{n}$ as $n \rightarrow \infty$ i.e. as n is larger and larger and larger?

2 Zeno's Paradox – Achilles and Tortoise

Zeno's Paradox of the Tortoise and Achilles

Source: http://www.mathacademy.com/pr/prime/articles/zeno_tort/



Zeno of Elea (circa 450 b.c.) is credited with creating several famous **paradoxes**, but by far the best known is the paradox of the Tortoise and Achilles. (Achilles was the great Greek hero of Homer's *The Illiad*.) It has inspired many writers and thinkers through the ages, notably Lewis Carroll and Douglas Hofstadter, who also wrote dialogues involving the Tortoise and Achilles.

The original goes something like this:

The Tortoise challenged Achilles to a race, claiming that he would win as long as Achilles gave him a small head start. Achilles laughed at this, for of course he was a mighty warrior and swift of foot, whereas the Tortoise was heavy and slow.

“How big a head start do you need?” he asked the Tortoise with a smile.

“Ten meters,” the latter replied.

Achilles laughed louder than ever. “You will surely lose, my friend, in that case,” he told the Tortoise, “but let us race, if you wish it.”

“On the contrary,” said the Tortoise, “I will win, and I can prove it to you by a simple argument.”

“Go on then,” Achilles replied, with less confidence than he felt before. He knew he was the superior athlete, but he also knew the Tortoise had the sharper wits, and he had lost many a bewildering argument with him before this.

“Suppose,” began the Tortoise, “that you give me a 10-meter head start. Would you say that you could cover that 10 meters between us very quickly?”

“Very quickly,” Achilles affirmed.

“And in that time, how far should I have gone, do you think?”

“Perhaps a meter – no more,” said Achilles after a moment's thought.

“Very well,” replied the Tortoise, “so now there is a meter between us. And you would catch up that distance very quickly?”

“Very quickly indeed!”

“And yet, in that time I shall have gone a little way farther, so that now you must catch that distance up, yes?”



“Ye-es,” said Achilles slowly.

“And while you are doing so, I shall have gone a little way farther, so that you must then catch up the new distance,” the Tortoise continued smoothly.

Achilles said nothing.

“And so you see, in each moment you must be catching up the distance between us, and yet I – at the same time – will be adding a new distance, however small, for you to catch up again.”

“Indeed, it must be so,” said Achilles wearily.

“And so you can never catch up,” the Tortoise concluded sympathetically.

“You are right, as always,” said Achilles sadly – and conceded the race.

For further discussion see:

http://en.wikipedia.org/wiki/Zeno's_paradoxes

3 Walking to the wall – geometric series formula

Zeno's Paradox may be rephrased as follows. Suppose I wish to cross the room. First, of course, I must cover half the distance. Then, I must cover half the remaining distance. Then, I must cover half the remaining distance. Then I must cover half the remaining distance . . . and so on forever. The consequence is that I can never get to the other side of the room.

What this actually does is to make *all* motion impossible, for before I can cover half the distance I must cover *half* of half the distance, and before I can do that I must cover half of half of half of the distance, and so on, so that in reality I can never move any distance at all, because doing so involves moving an infinite number of small intermediate distances first.

$$\frac{D}{2} + \frac{D}{4} + \frac{D}{8} + \frac{D}{16} + \dots = \frac{D}{2} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \right)$$

$$\text{Let } S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

$$\text{Then } S = 1 + \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \right)$$

$$S = 1 + \frac{1}{2} S$$

$$\Rightarrow S = 2$$

Geometric Formula:

$$S = 1 + q + q^2 + q^3 + q^4 + \dots$$

$$S = 1 + q \left(1 + q^2 + q^3 + q^4 + \dots \right)$$

$$S = 1 + qS$$

$$\Rightarrow S = \frac{1}{1-q}$$

4 Divergent Series

That was a neat trick. Let us try gain.

$$S = 1 + 2 + 4 + 8 + 16 + \dots$$

$$S = 1 + 2(1 + 2 + 4 + 8 + 16 + \dots)$$

$$S = 1 + 2S$$

$$\Rightarrow S = -\frac{1}{2} \quad \text{What ??????????}$$

Let us try this

$$S = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \dots$$

$$S = 1 - 1(1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \dots)$$

$$S = 1 - S$$

$$\Rightarrow S = \frac{1}{2} \quad \text{Weird but it makes sense(Euler)}$$

But Janice said (Who's Janice?)

Isn't

$$S = (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \dots$$

$$S = 0 + 0 + 0 + 0 + 0 + 0 + \dots = 0$$

I disagree said Robert. (Who's Robert?)

$$S = 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + \dots$$

$$S = 1 + 0 + 0 + 0 + 0 + 0 + \dots = 1$$

Who is right? Is this all a big nonsense?

When an anonymous Indian clerk send a letter to a famous British Mathematician G.H. Hardy, Hardy was impressed when he saw this equation

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \dots = -\frac{1}{12}$$

“He has found it! He knows!” Hardy said.

5 Continued Fractions

$$\sqrt{2} = \sqrt{2} - 1 + 1$$

$$\sqrt{2} = \frac{(\sqrt{2}-1)(\sqrt{2}+1)}{\sqrt{2}+1} + 1$$

$$\sqrt{2} = \frac{1}{\sqrt{2}+1} + 1$$

$$\sqrt{2} + 1 = \frac{1}{\sqrt{2}+1} + 2$$

Let $x = \sqrt{2} + 1$. Then

$$x = 2 + \frac{1}{x}$$

$$x = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}$$

Continued Fractions

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}$$

Question : can you do this for an arbitrary \sqrt{a} ?

Notice that

$$x = 2 + \frac{1}{x} \Leftrightarrow x^2 - 2x - 1 = 0$$

for $x \neq 0$ and that quadratic equation actually has two solutions $1 \pm \sqrt{2}$. What happened to $1 - \sqrt{2}$?

To obtain the continued fraction for $\sqrt{2}$ we could have started from the quadratic equation $x^2 - 2x - 1 = 0$. Then

$$x^2 = 1 + 2x \Rightarrow x = 2 + \frac{1}{x} .$$

From here continue as before. **Xplore!!**

6 Infinite Products

Euler

A consequence of fundamental theorem of Algebra is the following fact of life.

Every polynomial $P(x)$ can be written as a product of binomials $(x-r_j)$ where r_j are roots of the polynomial.

i.e.,

$$P(x) = C(x-r_1)(x-r_2)(x-r_3)(x-r_4)(x-r_5)(x-r_6)\cdots(x-r_n)$$

where C is a constant and r_j 's may repeat.

Euler imagined this is possible even for functions which are not polynomials.

$$\frac{\sin(x)}{x} = C \left(1 - \frac{x}{r_1}\right) \left(1 - \frac{x}{r_2}\right) \left(1 - \frac{x}{r_3}\right) \left(1 - \frac{x}{r_4}\right) \cdots$$

An infinite product.

Now zeros of $\frac{\sin(x)}{x}$ are $\pm n\pi$. Hence

$$\begin{aligned} \frac{\sin(x)}{x} &= C \left(1 - \frac{x}{\pi}\right) \left(1 + \frac{x}{\pi}\right) \left(1 - \frac{x}{2\pi}\right) \left(1 + \frac{x}{2\pi}\right) \left(1 - \frac{x}{3\pi}\right) \left(1 + \frac{x}{3\pi}\right) \left(1 - \frac{x}{4\pi}\right) \left(1 + \frac{x}{4\pi}\right) \cdots \\ &= C \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \left(1 - \frac{x^2}{16\pi^2}\right) \cdots \end{aligned}$$

Euler proved before that $\frac{\sin(0)}{0} = 1$ as $x \rightarrow 0$. Hence $C = 1$.

$$\frac{\sin(x)}{x} = \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \left(1 - \frac{x^2}{16\pi^2}\right) \cdots$$

when $x = \frac{\pi}{2}$

$$\frac{2}{\pi} = \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{16}\right) \cdots \left(1 - \frac{1}{4n^2}\right) \cdots$$

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right)$$

Viète

From

$$\sin(2x) = 2 \sin(x) \cos(x)$$

we have

$$\sin(x) = 2 \cos\left(\frac{x}{2}\right) \sin\left(\frac{x}{2}\right)$$

$$\sin(x) = 2 \cos\left(\frac{x}{2}\right) \sin\left(\frac{x}{2}\right) = 2^2 \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{4}\right) \sin\left(\frac{x}{4}\right) = 2^3 \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{4}\right) \cos\left(\frac{x}{8}\right) \sin\left(\frac{x}{8}\right)$$

and on and on and on

Finally

$$\sin(x) = 2^n \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{4}\right) \cos\left(\frac{x}{8}\right) \cdots \cos\left(\frac{x}{2^n}\right) \sin\left(\frac{x}{2^n}\right)$$

$$\frac{\sin(x)}{x} = \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{4}\right) \cos\left(\frac{x}{8}\right) \cdots \cos\left(\frac{x}{2^n}\right) \frac{\sin\left(\frac{x}{2^n}\right)}{\frac{x}{2^n}}$$

The last factor becomes 1 and we have

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} \cos\left(\frac{x}{2^n}\right)$$

In particular when $x = \frac{\pi}{4}$ we have Viète formula:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{2+\sqrt{2}}}{2} \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \cdots$$

Prove it.

7 Cantor's many Infinities

How many elements are in N ?

Cantor: I do not know, I guess ∞ .

How many elements are in the set $\{a, b, c\}$?

Alph. Student would answer _____?

Cantor: I do not know either but same as in $\{\otimes, \Delta, \nabla\}$ or in $\{1, 2, 3\}$.

Equipotent sets: Two sets have same number of elements (cardinality) if there is a bijection between two sets.

Notation: cardinality of a set $T := |T|$.

Shock: $|N| = |\text{Evens}|$.

Impossible?!

Also: $|N| = |Q|$.

Proof:

m/n	0	1	2	3	4
1	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$
2	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$
3	$\frac{0}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$
4	$\frac{0}{4}$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
5	$\frac{0}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$

Alp: I think that there are no sets which have more elements than N . Right?

Cantor: Not true.

$|R| > |N|$.

and I will prove it using my diagonal argument.

Diagonal argument to be inserted

But I also have a hunch or more like hypothesis

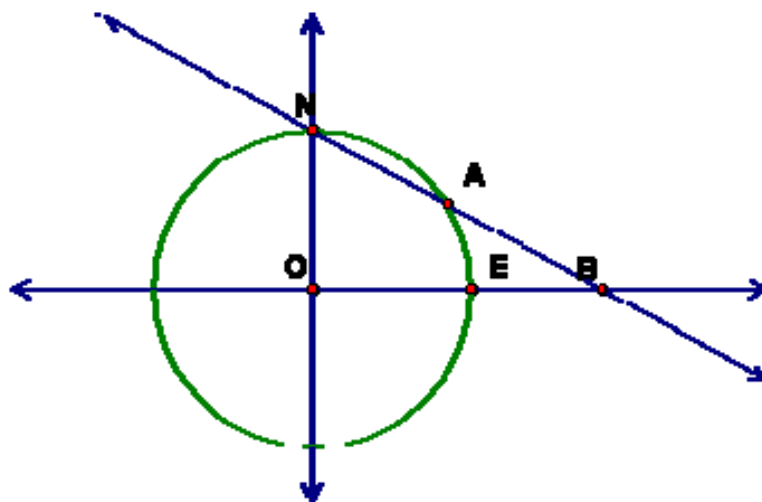
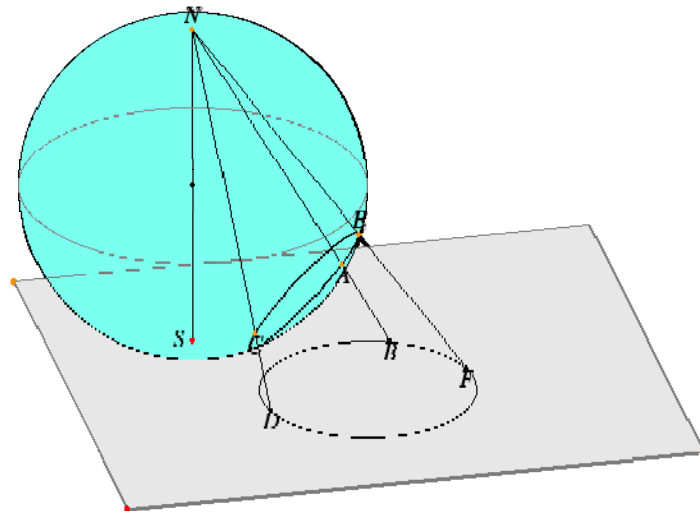
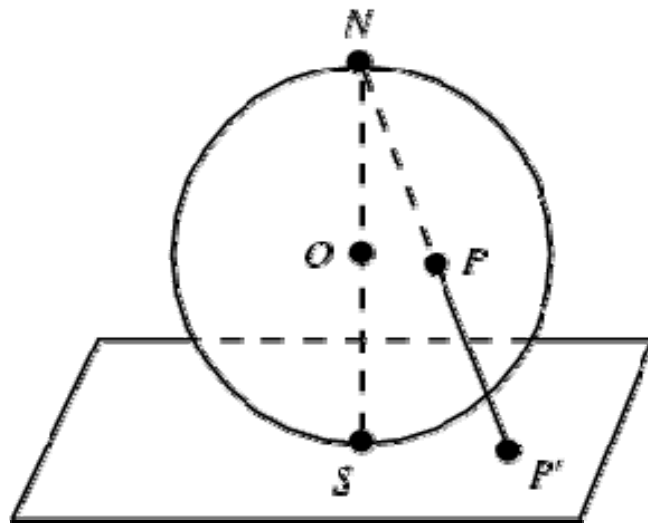
Continuum Hypothesis: There are no sets which cardinality is between $|\mathbb{R}|$ and cardinality of $|\mathbb{N}|$.

Notice: $|\mathbb{N}|$ is called Aleph zero \aleph_0 .

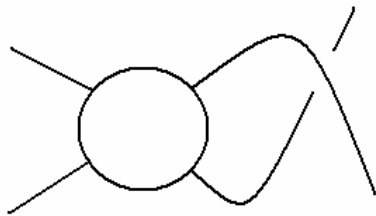
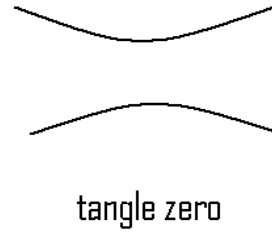
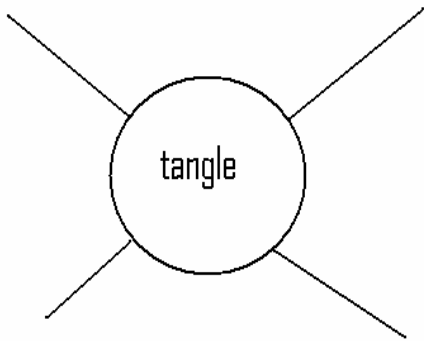
$|\mathbb{R}|$ is called continuum C .

8 Geometric perspective

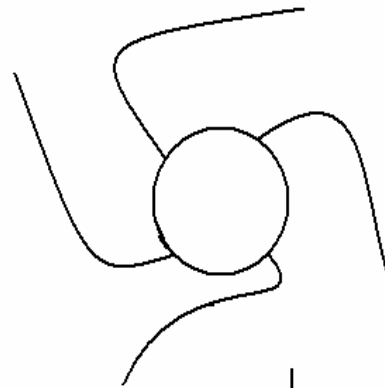
Sterographic Projection



Conway's Tangle Arithmetic



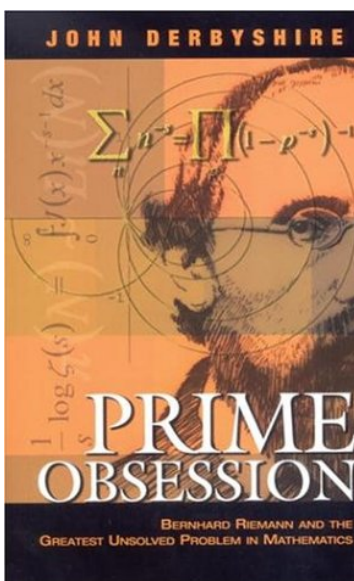
$t+1$



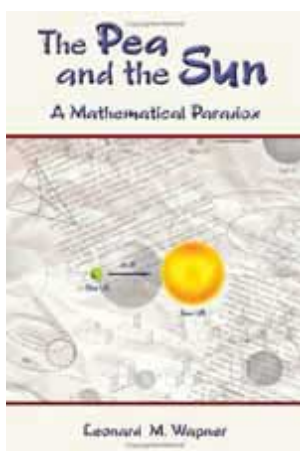
$-\frac{1}{t}$

9 Miscellaneous

Prime Obsession's Card Trick



Banach Tarski Paradox Pea and a sun



Consequences of the Banach–Tarski paradox are, for example:

An orange can be chopped into a finite number of chunks, and these chunks can then be put together again to yield two oranges, each of which has the same size as the one that just went into pieces.

Another, even more bizarre consequence is:

A pea can be split into a finite number of pieces, and these pieces can then be reassembled to yield a solid ball whose diameter is larger than the distance of the Earth to the sun.

<http://pims.math.ca/pi/issue2/page13-15.pdf>

Gabriel's Horn or Torricelli's trumpet

