

Geometric Sequence and Series

From Elementary School to Advance Research

Dr. Josip Derado
Kennesaw State University

To do Mathematics is to

Imagine !

Create!

Explore!

Elementary School

Ask students to multiply

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdots$$

How far can they go? Ask them to sum the results:

$$1 \quad 2 \quad 4 \quad 8 \quad 16 \quad 32 \quad 64$$

$$1 \quad 3 \quad 7 \quad 15 \quad 31 \quad 63 \quad ???$$

Can they guess what is going on? Hey can we do that for 3?

$$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdots$$

How far can they go now? Add the results:

$$1 \quad 3 \quad 9 \quad 27 \quad 81 \quad 243 \quad 729$$

$$1 \quad 4 \quad 13 \quad 40 \quad 121 \quad 364 \quad ???$$

Can you guess it know? No! Well multiply the sums by 3.

$$3 \quad 12 \quad 39 \quad 120 \quad 363 \quad 1092 \quad ???$$

Now, can you guess? No! Subtract the sums row from the last row.

$$2 \quad 8 \quad 26 \quad 80 \quad 242 \quad 728 \quad ???$$

Can, you guess now?!

It is MAagic

Elementary School

The example is an exercise in:

multiplication, addition drill

pattern recognition drill

opens field for further exploration

introduction into geometric sequences and series,

introduction into topic which is far advance for students, but...

Middle School: Do the same with fractions or negative numbers!

Middle/High School: Guess the formula! Discuss a validity of the formula!

Geometric Sequence and Sum

Geometric Sequence

Let $q \in \mathbb{R}$. Then

$$\begin{aligned}q = 1 &\Rightarrow q^n \rightarrow 1 \\q = -1 &\Rightarrow \text{has two partial limits } 1, -1 \\|q| < 1 &\Rightarrow q^n \rightarrow 0 \\q > 1 &\Rightarrow q^n \rightarrow \infty \\q < -1 &\Rightarrow q^n \text{ has two partial limits } \infty, -\infty\end{aligned}$$

Geometric sum

$$s_n = 1 + q + q^2 + q^3 + \cdots + q^n$$

Then (very much like in the elementary school exercise)

$$qs_n = q + q^2 + q^3 + \cdots + q^n + q^{n+1}$$

Subtract qs_n from s_n

$$\begin{aligned}s_n - qs_n &= (1 + q + q^2 + q^3 + \cdots + q^n) - (q + q^2 + q^3 + \cdots + q^n + q^{n+1}) \\(1 - q)s_n &= 1 - q^{n+1} \\s_n &= \frac{1 - q^{n+1}}{1 - q} \quad \text{Geometric sum formula - High School stuff}\end{aligned}$$

High School students can explore further the formula. For example what does happen when n is getting larger, and larger and larger.

Geometric Series

Let $s = 1 + q + q^2 + q^3 + \dots$. Then

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q}$$

The improper proof but a valid exercise for High School students:

$$\begin{aligned} s - qs &= (1 + q + q^2 + q^3 + \dots) - (q + q^2 + q^3 + \dots) \\ (1 - q)s &= 1 \\ s &= \frac{1}{1 - q} \quad \text{Cool stuff but totally incorrect!} \end{aligned}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \frac{1}{1 - \frac{1}{2}} = 2$$

Hey, a student from high school can claim then

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + \dots = \frac{1}{1 - 2} = -1$$

Crazy!, But this is wrong! Right? Right!

Geometric Series formula

$$1 + q + q^2 + q^3 + \cdots = \frac{1}{1 - q} \quad \text{providing } |q| < 1$$

Notice also

$$1 + q + q^2 + q^3 + \cdots + q^n < \frac{1}{1 - q}$$

Some Basic Applications of Geometric Sum and Series

Let z be a complex number such that

$$z^n = 1 \Rightarrow z = e^{i\frac{2\pi k}{n}} \quad k = 0, 1, 2, \dots, n-1 \quad \text{the roots of unity}$$

Compute

$$s_n = 1 + z + z^2 + z^3 + \dots + z^{n-1}.$$

Solution:

$$s_n = 1 + z + z^2 + z^3 + \dots + z^{n-1} = (\text{geometric sum}) = \frac{1 - z^n}{1 - z} = 0 \quad z \neq 1$$

Prove

$$\sum_{k=0}^{\infty} \frac{\sin(kx)}{r^k} = \frac{r \sin(x)}{1 + r^2 - 2r \cos(x)} \quad \text{for } r > 1.$$

Solution:

Recall

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\sin(kx)}{r^k} &= \sum_{k=0}^{\infty} \frac{e^{ikx} - e^{-ikx}}{2ir^k} \\ &= \frac{1}{2i} \sum_{k=0}^{\infty} \frac{e^{ikx}}{r^k} - \frac{e^{-ikx}}{r^k} \\ &= \frac{1}{2i} \sum_{k=0}^{\infty} \left(\frac{e^{ix}}{r}\right)^k - \left(\frac{e^{-ix}}{r}\right)^k \\ \text{geo. series} &= \frac{1}{2i} \left(\frac{1}{1 - \frac{e^{ix}}{r}} - \frac{1}{1 - \frac{e^{-ix}}{r}} \right) \\ &= \frac{1}{2i} \left(\frac{1 - \frac{e^{-ix}}{r}}{1 - \frac{e^{ix}}{r} - \frac{e^{-ix}}{r} + \frac{1}{r^2}} \right) \\ &= \frac{1}{2i} \frac{r(e^{ix} - e^{-ix})}{r^2 - r(e^{ix} + e^{-ix}) + 1} \\ &= \frac{r \sin(x)}{1 + r^2 - 2r \cos(x)} \end{aligned}$$

Geometric sums and series of Matrices

$$A^0 = I \quad A^n = \underbrace{AAAA \cdots A}_{n\text{-times}}$$

Compute

$$s_n = I + A + A^2 + A^3 + A^4 + \cdots + A^n = ?$$

Let us try the old elementary-high school trick:

$$As_n = A + A^2 + A^3 + A^4 + \cdots + A^{n+1}$$

Then

$$s_n - As_n = (I + A + A^2 + A^3 + A^4 + \cdots + A^n) - (A + A^2 + A^3 + A^4 + \cdots + A^{n+1})$$

$$(I - A)s_n = I - A^{n+1}$$

$$s_n = (I - A)^{-1}(I - A^{n+1}) \quad \text{Geometric sum formula for matrices}$$

But when does it work?

Geometric Sums and Series of Matrices

Matrix Norms:

$$\| AB \| \leq \| A \| \| B \| \quad \| A \| = 0 \Leftrightarrow A = 0$$

Then

$$\begin{aligned} \| A^n \| &\leq \| A \|^n \\ \| A^n \| &\rightarrow 0 \quad \text{providing} \quad \| A \| < 1 \end{aligned}$$

Hence we can show the following

$$\begin{aligned} \| s_{n+m} - s_n \| &= \| (I + A + A^2 + A^3 + A^4 + \dots + A^{n+m}) - (I + A + A^2 + A^3 + A^4 + \dots + A^n) \| \\ &= \| A^{n+1} + A^{n+2} + \dots + A^{n+m} \| \\ &\leq \| A^{n+1} \| + \| A^{n+2} \| + \dots + \| A^{n+m} \| \\ &\leq \| A \|^{n+1} (1 + \| A \| + \| A \|^2 + \dots + \| A \|^{m-1}) \\ &\leq \| A \|^{n+1} \frac{1}{1 - \| A \|} \rightarrow 0 \quad \text{since} \quad \| A^n \| \rightarrow 0 \end{aligned}$$

This means (s_n) is a Cauchy sequence. Hence the sequence converges. The following shows that $s_n \rightarrow (I - A)^{-1}$.

$$(I - A)s_n = (I - A)(I + A + A^2 + \dots + A^n) = I - A^{n+1}$$

Let $n \rightarrow \infty$

$$(I - A)s = I - \lim A^{n+1} = I$$

Hence

$$(I - A)^{-1} = s = I + A + A^2 + A^3 + \dots$$

Applications of Geometric Sum and Series in Calculus

D' Alambert ratio convergence test:

Let (a_n) be a sequence of real numbers such that

$$\limsup \frac{|a_{n+1}|}{|a_n|} = L < 1 \Rightarrow \sum_{n=0}^{\infty} a_n \text{ is convergent}$$

$$\limsup \frac{|a_{n+1}|}{|a_n|} = L > 1 \Rightarrow \sum_{n=0}^{\infty} a_n \text{ is divergent}$$

Essential part of the Proof:

$$\begin{array}{l} \exists r > 0, \quad N \in \mathbb{N} \quad \text{such that} \quad \frac{|a_{n+1}|}{|a_n|} < L < r < 1 \quad \forall n > N \\ a_{N+1} < a_N r \\ a_{N+2} < a_N r^2 \\ a_{N+3} < a_N r^3 \\ a_{N+4} < a_N r^4 \\ \vdots & \vdots & \vdots \end{array}$$

$$\sum_{k=0}^{\infty} a_{N+k} < a_N \sum_{k=0}^{\infty} r^k < \frac{a_N}{1-r}$$

Applications in Metric Spaces

Let (X, d) be a metric space.

Contraction Mapping Theorem

Let $f : X \rightarrow X$ such that

$$d(f(x), f(y)) \leq c d(x, y) \quad 0 < c < 1$$

Let

$$x_0 \in X, \quad x_{n+1} = f(x_n)$$

Then

$$f(x_n) \rightarrow \bar{x} \quad \text{and} \quad f(\bar{x}) = \bar{x}$$

Further \bar{x} is the unique fixed point.

Proof:

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq c d(x_n, x_{n-1}) \leq c^2 d(x_{n-1}, x_{n-2}) \leq \cdots \leq c^n d(x_1, x_0)$$

Hence

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, x_{n+m-1}) + d(x_{n+m-1}, x_{n+m-2}) + d(x_{n+m-2}, x_{n+m-3}) + \cdots + d(x_{n+1}, x_n) \\ &\leq c^{n+m-1} d(x_1, x_0) + c^{n+m-2} d(x_1, x_0) + \cdots + c^n d(x_1, x_0) \\ &\leq c^n d(x_1, x_0) (1 + c + c^2 + c^3 + \cdots + c^{m-1}) \\ &\leq c^n d(x_1, x_0) \frac{1}{1-c} \rightarrow 0 \end{aligned}$$

(x_n) is a Cauchy sequence $\Rightarrow (x_n)$ is convergent to \bar{x} .

Next we show that \bar{x} is a fixed point.

$$\begin{aligned}d(f(\bar{x}), \bar{x}) &= d(f(\bar{x}), \lim x_n) \\&= \lim d(f(\bar{x}), x_n) \\&= \lim d(f(\bar{x}), f(x_n)) \\&\leq \lim c d(\bar{x}, x_n) \rightarrow 0\end{aligned}$$

Contraction Mappings in Calculus

Let $f : I \rightarrow I$ is differentiable on I then by MVT

$$|f(x) - f(y)| \leq |f'(\xi)| |x - y| \leq M|x - y|$$

where

$$M = \sup_I |f'(\xi)|$$

So if $M < 1$ f is a contraction mapping.

Example:

Solve $\cos(x) - x = 0$.

Create the recursive sequence

$$x_n = \cos(x_{n-1})$$

$$\left| \frac{d}{dx} \cos(x) \right| = |-\sin(x)| \leq 1$$

So if we stay far away from the 0 we would be o.k.

Von Neuman Series

Let $(X, \|\cdot\|)$ be a Banach Space.

Let $T : X \rightarrow X$ be a bounded linear operator and let

$$\|T\| < \lambda$$

Then

$$T_\lambda = (T - \lambda I)^{-1} = - \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} T^n$$

and

$$\|T_\lambda\| \leq \frac{1}{\lambda - \|T\|}$$

Proof:

$$\begin{aligned} \|T\| < \lambda &\Rightarrow \left\| \frac{T}{\lambda} \right\| < 1 \\ \sum_{n=0}^{\infty} \left\| \frac{T^n}{\lambda^n} \right\| &\leq \sum_{n=0}^{\infty} \left\| \frac{T}{\lambda} \right\|^n < \infty \end{aligned}$$

$$\begin{aligned} (T - \lambda I)B &= (T - \lambda I) \left(\sum_{n=0}^{\infty} \frac{T^n}{\lambda^n} \right) \\ &= \sum_{n=0}^{\infty} (T - \lambda I) \frac{T^n}{\lambda^n} \\ &= \sum_{n=0}^{\infty} \frac{T^{n+1}}{\lambda^n} - \frac{T^n}{\lambda^{n-1}} \\ &= \lambda \sum_{n=0}^{\infty} \frac{T^{n+1}}{\lambda^{n+1}} - \frac{T^n}{\lambda^n} \\ &= -\lambda I \end{aligned}$$

$$\Rightarrow \frac{B}{-\lambda} = (T - \lambda I)^{-1}$$
$$(T - \lambda I)^{-1} = - \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} T^n$$

Divergent Series

Hardy and Ramanujan

$$1 + 2 + 3 + 4 + 5 + 6 + \dots = -\frac{1}{12}$$

Imagine: We can do it!

Create!!

Suppose that

$$(1) \quad \sum a_n = s \Rightarrow \sum ka_n = k \sum a_n = ks$$

$$(2) \quad \sum a_n = s \quad \sum b_n = t \Rightarrow \sum a_n + b_n = t + s$$

$$(3) \quad a_0 + a_1 + a_2 + a_3 + \dots = s \Rightarrow a_1 + a_2 + a_3 + \dots = s - a_0$$

Explore!

$$1 + (-1) + 1 + (-1) + 1 + (-1) + \dots = 1 - (1 + (-1) + 1 + (-1) + \dots)$$

$$s = 1 - s \Rightarrow s = \frac{1}{2}$$

Actually

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + \dots = -1$$

(:O)

At the End

Why this talk?

Importance of Geometric sequences, sums, and series in mathematics

Message to the teachers - Be a PUFM teacher