

On TMRAs and Multiwavelet sets

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Notation:

$A \equiv$ an integer valued expanding dilation

$$B = A^* \quad q = \det(A)$$

$$D = \{d_0 = e, d_1, d_2, \dots, d_{q-1}\} = \frac{\mathbb{Z}^d}{A\mathbb{Z}^d}$$

$$L = \{l_0 = e, l_1, l_2, \dots, l_{q-1}\} = \frac{\mathbb{Z}^d}{B\mathbb{Z}^d}$$

$$G = \{g_0 = e, g_1, g_2, \dots, g_{q-1}\} = \frac{B^{-1}\mathbb{Z}^d}{\mathbb{Z}^d}$$

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi x \cdot \xi} dx$$

Madych (1993)Some Elementary Properties of MRA of $L^2(\mathbb{R}^d)$

Definition 1. An MRA (Multiresolution Analysis) \mathcal{V} with the basic space V_0 is TMRA (Translation Multiresolution Analysis) if

$$f(x) \in V_0 \Rightarrow f(x - y) \in V_0 \quad \forall y \in \mathbb{R}^n$$

Theorem 1. Suppose \mathcal{V} is a TMRA $L^2(\mathbb{R}^d)$ associated with (\mathbb{Z}^d, A) and $B = A^*$. Then $\widehat{V}^0 = L^2(\Omega)$ where Ω is a closed subset of \mathbb{R}^d with the following properties:

- (T1) $\Omega \subseteq B\Omega$
- (T2) $\Omega \cap (\Omega + k) \simeq 0 \quad \forall k \in \mathbb{Z}^d$
- (T3) $\cup_{k \in \mathbb{Z}^d} (\Omega + k) = \mathbb{R}^d$
- (T4) $\cup_{j \in \mathbb{Z}^d} (B^j \Omega) = \mathbb{R}^d$

Conversely, if $\widehat{V}_j := L^2(B^j \Omega)$ where Ω is a closed subset of \mathbb{R}^d with properties (T1)-(T4) then the $\{V_j\}_{j \in \mathbb{Z}}$ is TMRA of $L^2(\mathbb{R})$

Remarks:

- $(T1) \implies 0 \in \Omega$
- $[-\epsilon, \epsilon]^d \subseteq \Omega \implies (T4)$
- Ω is a scaling set.
- $B\Omega \setminus \Omega$ is called a multiwavelet set.

Definition 2. A set $W \subset \mathbb{R}^d$ is a multiwavelet set of order L if $W = \bigcup_{l=1}^L W_l$ satisfying

$$\sum_{k \in \mathbb{Z}^d} \chi_{W_l}(\xi + k) \chi_{W_{l'}}(\xi + k) = \delta_{l,l'} \quad a.e. \quad \forall l, l' = 1, \dots, L$$

$$\sum_{j \in \mathbb{Z}} \sum_{l=1}^L \chi_{W_l}(B^j \xi) = 1$$

TMRA case

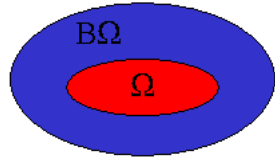


Figure 1: Scaling set

$$W_j = \Omega \cap (\Omega + l_j + B\mathbb{Z}^d)$$

where $l_j \in L$ are coset representatives of $\frac{\mathbb{Z}^d}{B\mathbb{Z}^d}$.
 $B^{-1}\Omega + \mathbb{Z}^d$ is a symbol set.

Literature-Historical list

- **1993 Madych** Some elementary properties of MRA
- **1996 Hernandez, Weiss, Wang** Smoothing MSF wavelets(Part I and II)
- **1996 Fang, Wang** Construction of MSF wavelets
- **1997 Dai, Larson,Speegle** Wavelet sets in \mathbb{R}^n Part I and II
- **1997 Ionascu, Pearcy** On subwavelet sets
- **1998 Ionascu, Larson, Pearcy**On wavelet sets
- **1998 Soardi, Weiland**Single Wavelets in n-dimensions
- **1999 Bagett, Medina, Merill** GMRA and a construction procedure for all wavelet sets in \mathbb{R}^n
- **1999 Benedetto, Leon** The construction of multiple dyadic MSF wavelets
- **2000 Dai, Diao, Gu** Frame wavelet sets in \mathbb{R}^n
- **2000 Benedetto, Sumatkijakan**
- **2000 Bownik, Rzeszotnik, Speegle**

Elementary Examples

Shannon Wavelet ($A = [2]$)

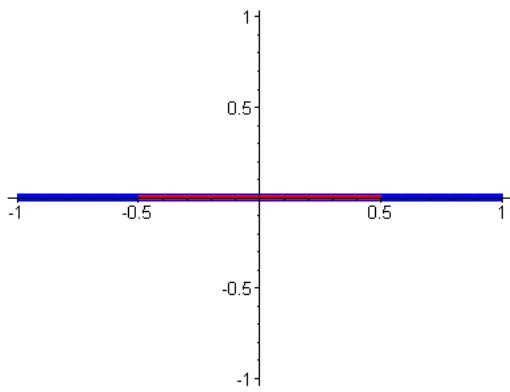


Figure 2: Shannon Wavelet

$$\phi(x) = \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \quad \text{scaling function}$$
$$\psi(x) = \frac{\sin(2\pi x) - \sin(\pi x)}{\pi x} = \frac{\sin(\pi x)}{\pi x} (2\cos(\pi x) - 1) \quad \text{wavelet}$$

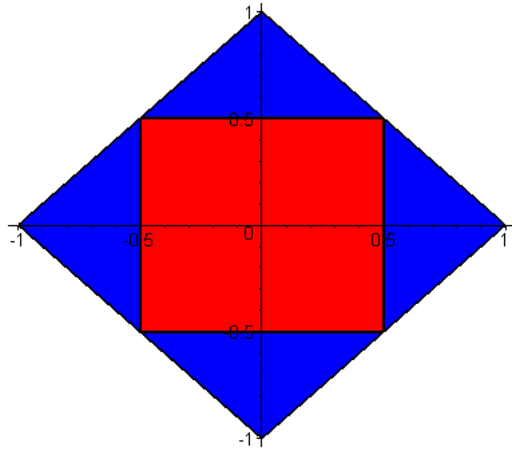


Figure 3: Diamond Wavelet

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\phi(x, y) = \text{sinc}(x)\text{sinc}(y) \quad \text{scaling function}$$

$$\psi(x, y) = 2\text{sinc}(x + y)\text{sinc}(y - x) - \text{sinc}(x)\text{sinc}(y) \quad \text{wavelet}$$

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

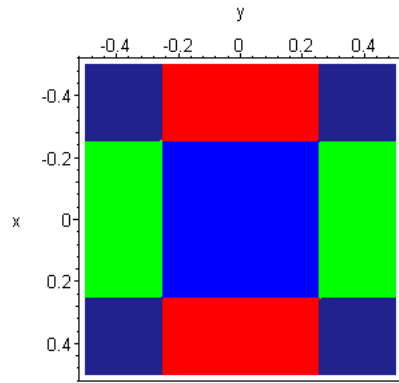


Figure 4: 2I case

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

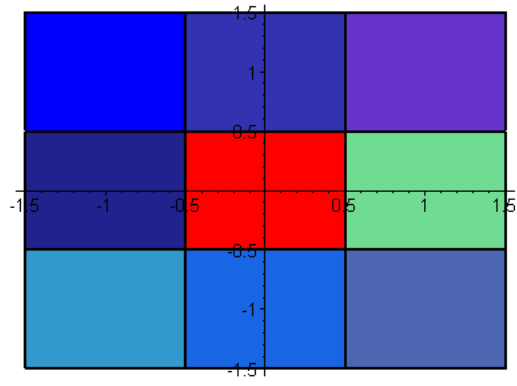


Figure 5: 3I case

Self-similar scaling sets

$B\Omega = \bigcup_j \Omega + d_j$ where $d_j \in D$ and $\Omega + d_j$ are disjoint

$$\begin{aligned}\phi(x) &= \int_{\Omega} e^{2\pi i x \cdot \xi} d\xi \\ &= \sum_j \int_{B^{-1}\Omega + B^{-1}d_j} e^{2\pi i x \cdot \xi} d\xi \\ &= \sum_j \frac{1}{q} \int_{\Omega} e^{2\pi i x \cdot B^{-1}u} e^{2\pi i x \cdot B^{-1}d_j} du \\ &= \underbrace{\sum_j \frac{1}{q} e^{2\pi i A^{-1}x \cdot d_j}}_{m(A^{-1}x)} \underbrace{\int_{\Omega} e^{2\pi i x \cdot B^{-1}u} du}_{\phi(A^{-1}x)} \\ \Rightarrow \phi(x) &= \prod_{j \geq 1} m(A^{-j}x) \\ \Rightarrow \phi(A^{-1}k) &= \prod_{j \geq 2} m(A^{-j}k)\end{aligned}$$

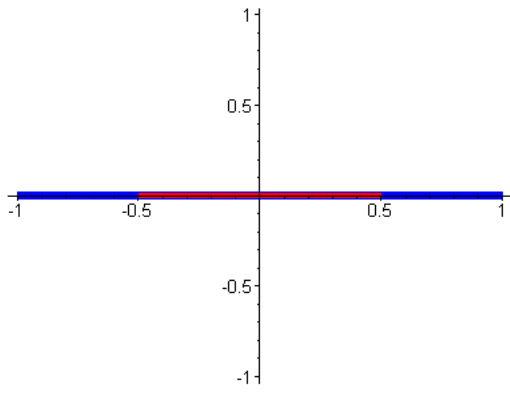


Figure 6: Shannon Wavelet

In particular

$$2\Omega = \left(\Omega + \frac{1}{2}\right) \cup \left(\Omega - \frac{1}{2}\right)$$

Notice $\frac{1}{2}$ and $-\frac{1}{2}$ are not digits.

$$\begin{aligned} \phi(2x) &= \frac{1}{2}(e^{-2\pi i \frac{x}{2}} + e^{2\pi i \frac{x}{2}})\phi(x) \\ &= \cos(\pi x)\phi(x) \end{aligned}$$

$$\phi(x) = \frac{\sin(\pi x)}{\pi x} = \prod_{j \geq 1} \cos\left(\frac{\pi x}{2^j}\right) \quad \text{Euler - Viete formula}$$

If

$$B\Omega = \bigcup_j \Omega + l_j$$

where l_j are digits then wavelets are given by

$$\psi_j(x) = e^{2\pi i l_j x} \phi(x) = e^{2\pi i l_j x} \prod_{j \geq 1} m(A^{-j} x)$$

A simple construction of TMRA
A general algorithm from MRA to TMRA

$$\begin{array}{c} m_0(\xi) \\ \text{a symbol for MRA(ONB)} \\ \Downarrow \\ M(\xi) := |m_0(\xi)|^2 \\ \Downarrow \\ K + \mathbb{Z}^d := \{\xi \in \mathbb{R}^d \mid M(\xi) = \max\{M(\xi), M(\xi + g_1), \dots, M(\xi + g_{q-1})\}\} \\ \Downarrow \\ \Omega := \bigcap_{j \geq 1} B^j K + B^j \mathbb{Z}^d \\ \Downarrow \\ \text{(providing } K \subset BK) \Rightarrow \Omega = BK \end{array}$$

Remarks:

- Choose K such that $K \subset BK$
- A possible choice for a starting symbol $m_0(\xi) := \frac{1}{q} \sum_{d_j \in D} e^{-2\pi i d_j \xi}$

Gröchenig, Haas, Madych
Haar-type scaling functions, $AT = \bigcup_j T + d_j$

- $M(\xi)$ is a filter for a biorthogonal(interpolating) scaling function

$$\begin{aligned} M(0) &= 1 \\ \sum_{g \in G} M(\xi + g) &= 1 \end{aligned}$$

Let

$$\Delta^q = \{x \in \mathbb{R}_+^q \mid \sum_{j=1}^q x_j = 1\}$$

For example $q = 3$

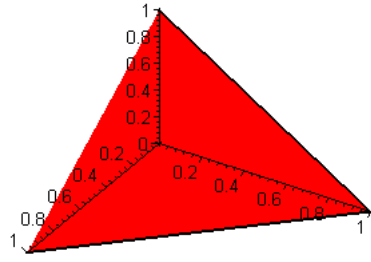


Figure 7: Δ^3

$$G = \frac{B^{-1}\mathbb{Z}^d}{\mathbb{Z}^d} = \{e, g, h\}$$

identify

$$e \leftrightarrow 0$$

$$g \leftrightarrow 1$$

$$h \leftrightarrow 2$$

$$(x_0, x_1, x_2) = (x_e, x_g, x_h)$$

For $g' \in G$ define

$$\sigma_{g'}(x_0, x_1, x_2) = \sigma_{g'}(x_e, x_g, x_h) := (x_{e+g'}, x_{g+g'}, x_{h+g'})$$

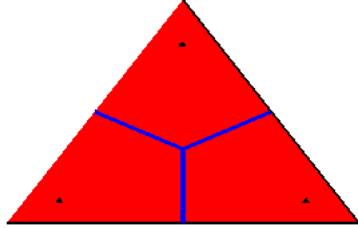


Figure 8:

Define $f : \Delta^q \rightarrow [0, 1]$ such that

- (1) $f(1, 0, 0) = 1$
- (2) $\sum_{g \in G} f(\sigma_g x) = 1 \quad \forall x \in \Delta^q$

Theorem 2. *Let M and f be defined as before and let f be sufficiently "nice" Then*

$$f(M(\xi), M(\xi + g), M(\xi + h))$$

is a symbol for an (biorthogonal)interpolating scaling function

Remark: 1-dimension Lemarie-Rieusset

Hence using the general algorithm we can construct variety of TMRAs.

Examples

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

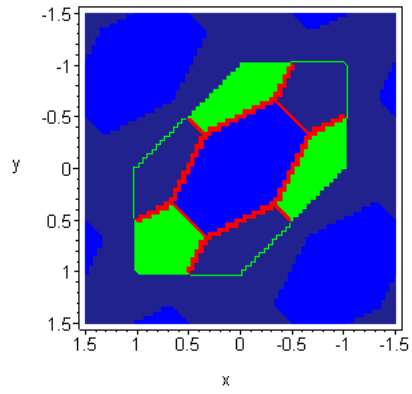


Figure 9: Hexagonal Wavelet

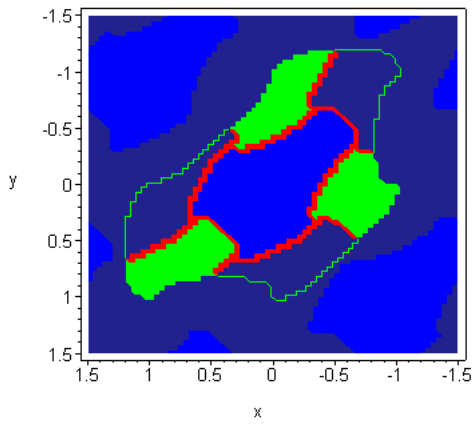


Figure 10: Oblique Hexagonal Wavelet

How hard is to construct f

First find \bar{f} such that

$$\begin{aligned}\bar{f}(1, 0, 0) &= 1 \\ \bar{f}(0, 1, 0) &= \bar{f}(0, 0, 1) = 0 \\ \bar{f}(x) &\geq 0 \\ \sum_{g \in G} \bar{f}(\sigma_g x) &> 0\end{aligned}$$

Then define

$$f(x) = \frac{\bar{f}(x)}{\sum_{g \in G} \bar{f}(\sigma_g x)}$$

This is not the end of the story!

Let

$$M_{P,f}(\xi) := B_P(f)\left(M(\xi), M(\xi + g), M(\xi + h)\right)$$

$$M_{P,f}(\xi) := \sum_{k \in I} f\left(\frac{k}{P}\right) \binom{P}{k} [M(\xi)]^{k_0} [M(\xi + g)]^{k_1} [M(\xi + h)]^{k_2}$$

$$I := \{k = (k_0, k_1, k_2) \in \mathbb{N}^3 \mid k_0 + k_1 + k_2 = P\}$$

B_P - Bernstein polynomials on the simplex Δ^3

Theorem 3. $M_{P,f}$ are symbols for an interpolating scaling functions.

Moreover, $M_{P,f}$ are trigonometric polynomials, and symbols for **nonseparable, compactly supported interpolating scaling functions.**

Finale

$$\begin{array}{llll} m_0(\xi) & \rightarrow \sqrt{B_2}f \rightarrow \sqrt{B_3}f \rightarrow \dots & \text{F-scaling functions(ONB)} & \\ \downarrow & \uparrow \sqrt{} & \uparrow \sqrt{} & \\ M(\xi) & \rightarrow B_2f \rightarrow B_3f \rightarrow \dots & f(M(\xi), M(\xi + g), M(\xi + h)) & \\ \downarrow & \text{by general algorithm} & \downarrow & \\ \Omega_0 & \rightarrow \Omega_{2,f} \rightarrow \Omega_{3,f} \rightarrow \dots & \Omega_{\infty,f} & \end{array}$$

$$M_{P,f,t} := tM_{P-1,f} + (1-t)M_{P,f} \quad t \in [0, 1]$$

Use the general algorithm to create a chain of TMRAs."morphing" of MRAs or TMRAs

Special Example

$$E := \{x \in \Delta^3 \mid x_0 = \max(x_0, x_1, x_2)\}$$

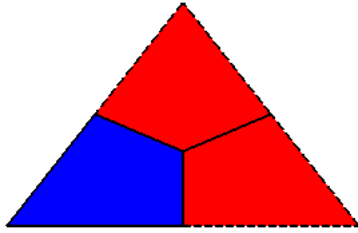


Figure 11: Starting set

$$f := \chi_E$$

$$M_{P,f} \rightarrow \phi_{P,f}$$

$\phi_{P,f} \Rightarrow$ increasing order of vanishing moments (Strang-Fix order) and increasing regularity.
TMRA stays unchanged throughout the whole chain.

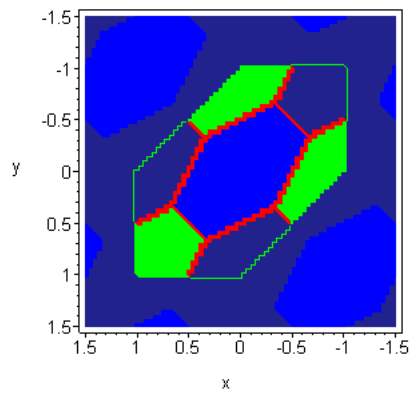


Figure 12: Hexagonal Wavelet

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