Recall:
The Laplace transform is defined as
\[
\mathcal{L}[f](s) := \int_0^\infty e^{-st}f(t)dt
\]

In class we proved the following properties of the Laplace Transform:
(a) \(\mathcal{L}[f + g] = \mathcal{L}[f] + \mathcal{L}[g]\).
(b) \(\mathcal{L}[c \cdot f] = c \cdot \mathcal{L}[f], \ c \in \mathbb{R}\).
(c) \(\mathcal{L}[y'] = s \cdot \mathcal{L}[y] - y(0)\).

1. Problem - 6pts

(a) Use only property (c) of the Laplace Transform (see above) to compute the Laplace transform of \(f(t) = \cos(at)\), \(a \in \mathbb{R}\). DO NOT INTEGRATE!
(b) Compute the Laplace transform of \(tsin(t)\).

2. Problem - 6pts

Gamma function is defined as:
\[
\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt
\]

In class we proved:
\(\Gamma(x + 1) = x\Gamma(x) \quad x > 0\)

Prove the following identity:
\[
\int_0^1 [\ln(x)]^n dx = (-1)^n\Gamma(n + 1) \quad \forall n > 0
\]

Hint: Use substitution \(t = -\ln(x)\).

Group B

3. Problem - 4pts

Compute the Laplace transform of \(\cos(at - b)\) \(a, b \in \mathbb{R}\).
4 Problem - 4pts

Prove the following identity:

\[ \int_0^\infty e^{-x^n} \, dx = \Gamma \left( \frac{1}{n} + 1 \right) \quad \forall n > 0 \]

Hint: Use substitution \( t = x^n \).

5 Problem - 4pts

Recall

\[ \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \]

Use it to compute:

\[ \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2s^2}} \, dx \quad \forall m, s \in \mathbb{R} \]

Group C

6 Problem - 3pts

Compute

\[ \int_0^\infty r e^r \, dr \]

7 Problem - 3pts

Compute the Laplace transform of \( y(t) = t^2 + \cos(t) \).

8 Problem - 3pts

Let

\[ f(x) := \begin{cases} 
\frac{1}{\sqrt{t \cdot \ln^2(t)}} & t > 2 \\
0 & t \leq 2 
\end{cases} \]

Compute \( \|f(x)\| \).

9 Problem - 3pts

Recall

\[ \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \]

Use it to compute:

\[ \int_0^\infty e^{-\frac{x^2}{4}} \, dx \]