All the problems are 3 points each. Recall

1. **The Fact of Life of sequences**

   Every bounded and monotone sequence is convergent.

**Group A**

**Problem 1**

Let \( a_n = n^{\frac{1}{n}} \).

(a) Is \( (a_n) \) bounded? Prove or disprove.

(b) Is \( (a_n) \) monotone? Prove or disprove.

(c) Is \( (a_n) \) convergent? Prove or disprove. If it is find its limit.

**Problem 2**

A sequence is defined as \( a_n = \frac{n!}{n^n} \).

Is the sequence convergent or divergent? If it is prove the convergence using The Fact of Life Theorem or The Squeeze theorem and find the limit. If it is not explain why not.

**Problem 3**

Let \( (a_n) \) be defined as

\[
a_1 = 2, \quad a_{n+1} = \frac{1}{2} \left( a_n + \frac{2}{a_n} \right)
\]

Prove that \( (a_n) \) is convergent and find its limit.

**Problem 4**

Let \( (a_n) \) be a sequence which converges to 0, and let \( (b_n) \) be a sequence which is bounded, but not necessarily convergent. Prove that \( \lim a_n b_n = 0 \).
Group B

Problem 1

Are the following sequences bounded? If they are, find an upper bound, and a lower bound. Prove the bounds really are the bounds.

(a) \( \frac{1}{n+1} - \frac{1}{n} \)

(b) \( \frac{1}{2^n} - 3^n \)

Problem 2

Are the following sequences convergent or not. Prove the convergence using The Fact of Life Theorem or The Squeeze theorem. If they are find the limit, if they are not explain why not.

(a) \( \frac{\sin^2(n)}{2^n} \)

(c) \( \sqrt{\frac{2n}{n+1}} \)

Problem 3

Let \( (a_n) \) be defined as

\[ a_1 = 0, \quad a_{n+1} = \sqrt{5 + a_n} \]

Prove that \( (a_n) \) is convergent and find its limit.

Problem 4

Let \( (a_n) \) be defined as

\[ a_1 = 1, \quad a_{n+1} = \frac{1}{2} \left( a_n + \frac{p}{a_n} \right) \]

where \( p > 0 \). The sequence is convergent. You need only to find its limit.

Group C

Problem 1

Are the following sequences bounded? If they are, find an upper bound, and a lower bound.

(a) \( \frac{1}{1+n^2} \)

(b) \( \frac{1}{n} - n^2 + 3n - 1 \)

(c) \( \frac{1}{5^n} n! \)
Problem 2

Are the following sequences convergent or not. If they are find the limit, if they are not explain why not.

(a) \( \frac{n^2 - n + 1}{1 - 3n^2} \)

(b) \( \frac{\ln(n)}{n} \)

(c) \( \frac{9^n}{n!} \)

Problem 3

Let \( (a_n) \) be defined as

\[ a_0 = 1, \quad a_{n+1} = 2 + \frac{1}{3}a_n \]

Prove that \( (a_n) \) is convergent and find its limit.

Problem 4

Let \( (a_n) \) be defined as

\[ a_1 = 1, \quad a_{n+1} = \frac{1}{2} \left( a_n + \frac{7}{a_n} \right) \]

Use Excel or some other software and list the first 20 elements of the sequence. Find its limit (exact limit not decimal approximation).