Integration by Parts

1 Problem
Using the integration by parts compute
\[ \int x^3 e^{-x} \, dx. \]

2 Problem
Using the integration by parts compute
\[ (a) \int e^{-2t} \cos(100t) \, dt. \]
\[ (b) \int e^{-at} \cos(bt) \, dt. \]

3 Problem
Using the integration by parts compute
\[ (a) \int \arctan(5t) \, dt. \]
\[ (b) \int \arctan(\lambda t) \, dt. \]

Improper integrals

4 Problem
Compute
\[ \int_0^\infty \frac{1}{1 + x^2} \, dx. \]

5 Problem
Compute
\[ \int_0^\infty e^{-t} \sin t \, dt. \]
6 Problem

Compute

\[ \int_2^\infty \frac{1}{x\ln(x)} \, dt \]

7 Problem

Find the value of \( a \) such that

\[ \int_0^\infty te^{-at} \, dt = 1 \]

Norm

We defined the Euclidean norm of a function \( f(x) \) as

\[ ||f|| := \int_{-\infty}^{\infty} f^2(x) \, dx. \]

The Euclidean norm has the following properties:

(a) \( ||f|| \geq 0 \).
(b) \( ||cf|| = |c|||f|| \quad \forall \ c \in \mathbb{R} \).
(c) \( ||f + g|| \leq ||f|| + ||g|| \)
(d) \( |\int_{-\infty}^{\infty} f(x)g(x) \, dx| \leq ||f|| \cdot ||g|| \). **Schwarz-Cauchy-Bunjakowski inequality**

8 Problem

Let

\[ f(x) = \begin{cases} 
xe^{-x^2} & x \geq 0 \\
0 & x < 0 
\end{cases} \]

Compute \( ||f|| \).

9 Problem

Let

\[ f(x) = \begin{cases} 
x + 1 & -1 \leq x \leq 0 \\
1 - x & 0 \leq x \leq 1 \\
0 & \text{otherwise} 
\end{cases} \]

Compute \( ||f|| \).

10 Problem

Let \( |f(x)| \leq |g(x)| \) for all \( x \in \mathbb{R} \).
(a) Show that

\[ ||f|| \leq ||g|| \]
(b) Let

\[ f(x) = \begin{cases} \frac{\sin(x)}{x^2} & x \geq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases} \]

Use (a) to prove that \[ \|f\| \leq 1 \]

11 Problem

Let \( f_k(x) \) be a collection of functions such that \( \|f_k\| < \frac{1}{2k} \).

(a) Show that \[ \|f_1 + f_2 + f_3 + \cdots + f_N\| \leq 1 \quad \forall \ N \in \mathbb{N} \]

(b) Let \( c_k \) be a sequence of real numbers such that \( |c_k| < A \) where \( A \) is a positive real number. Show that \[ \|c_1f_1 + c_2f_2 + c_3f_3 + \cdots + c_Nf_N\| \leq A \quad \forall \ N \in \mathbb{N} \]

Laplace Transform

The Laplace transform is defined as

\[ \mathcal{L}[f](s) := \int_0^\infty e^{-st} f(t)dt \]

12 Problem

Compute the Laplace transform of \( f(x) = \sin(ct) \).

13 Problem

Compute the Laplace transform of \( f(x) = e^{-ct} \).

14 Problem

Let

\[ H(x) = \begin{cases} 3 & x \geq 0 \\ 0 & x < 0 \end{cases} \]

Compute the Laplace transform of \( H(x) \).