Problem 1
Let \(A\) and \(B\) are 5x5 matrices, and let \(\text{det}(A) = 5\) and \(\text{det}(B) = 3\). Find \(\text{det}(2A^{-1}B^T)\).

Solution
\[
\text{det}(2A^{-1}B^T) = 2^5 \text{det}(A^{-1}B^T) = (\text{Binet – Cauchy}) = 32 \text{det}(A^{-1}) \text{det}(B^T) = 32 \frac{1}{\text{det}(A)} \text{det}(B) = \frac{96}{5}
\]

Problem 2.
(a) Let \(A\) be a 2x2 matrix and \(u\) and \(v\) two dimensional vectors. Show that
\[
u \cdot (Av) = (A^Tu) \cdot v
\]
(b) Is the identity true in higher dimensions? If it is, can you prove it, if it is not, can you find a counterexample. Hint: Remember that \(u \cdot v = u^Tv\).

Solution (a)
\[
u \cdot (Av) = (Av) \cdot u = (Av)^Tu = v^TA^Tu = v^T(A^Tu) = (A^Tu) \cdot v
\]
(b) Yes. The proof in (a) works for any dimension.

Problem 3. Find the angle between two main diagonals of a cube? Does the angle depend on the size of the cube? Solution Let \(k\) be the length of a side of a cube. Position your cube so that one vertex falls into origin and threes sides go along three axes. Then one main diagonal \(e\) is determined by the vertices \((0,0,0)\) and \((k,k,k)\) and for the other one \(f\) you can choose one which is determined by vertices \((0,k,0)\) and \((k,0,k)\). Hence \(e = (k,k,k)\) and \(f = (k,-k,k)\). The angle \(\alpha\) between them is computed as follows
\[
\alpha = \arccos \frac{e \cdot f}{||e|| ||f||} = \arccos \frac{k^2}{3k^2} = \arccos \frac{1}{3}
\]
Final result does not depend on the size of the cube.

Problem 4. Let \(u = (1, 2, -1)\) and \(b = (1, 1, 0)\).
(a) Find the orthogonal projection of \(u\) onto \(b\) and its length.
(b) Find the vector component of \(u\) orthogonal to \(b\) and its length.
(c) Find a vector perpendicular to \(u\) and \(v\) such that its length is 1. Solution:
(a)
\[
\text{proj}_b(u) = \frac{u \cdot b}{||b||^2} b = \left(\frac{3}{2}, \frac{3}{2}, 0\right)
\]
\[ \|\text{proj}_b(u)\| = \|\left(\frac{3}{2}, \frac{3}{2}, 0\right)\| = \sqrt{\frac{9}{2}} \]

(b) \[ \text{proj}^\perp_b(u) = u - \text{proj}_b(u) = (1, 2, -1) - \left(\frac{3}{2}, \frac{3}{2}, 0\right) = \left(-\frac{1}{2}, \frac{1}{2}, -1\right) \]

\[ \|\text{proj}^\perp_b(u)\| = \|\left(-\frac{1}{2}, \frac{1}{2}, -1\right)\| = \sqrt{\frac{3}{2}} \]

(c) Denote the vector we are looking for by \( w \). Then

\[ w = \frac{\mathbf{u} \times \mathbf{b}}{\|\mathbf{u} \times \mathbf{b}\|} = \frac{(1, -1, -1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \]

**Problem 5.** Let \( \mathbf{u}_1 = (1, -1, 1, -1) \), \( \mathbf{u}_2 = (2, 1, 2, 1) \), \( \mathbf{u}_3 = (1, 0, 3, 0) \). Find the vector which is orthogonal to \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \).

**Solution** Let \( i = (1, 0, 0, 0) \), \( j = (0, 1, 0, 0) \), \( k = (0, 0, 1, 0) \), and \( l = (0, 0, 0, 1) \). Then the vector \( w \) which is perpendicular to all \( \mathbf{u}_j \), \( j = 1, 3 \) is given by the formula:

\[ w = \frac{i}{1} \begin{vmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 1 & 0 & 3 & 0 \end{vmatrix} = -6j + 6k = (0, -6, 0, 6) \]