1 Group A

Problem 1.(3pts)

Prove the following statement.

**Statement:** Let \( m \) be a composite number. Then there exists a prime number \( p \) such that \( p|m \).

Recall:

An integer is a composite number if it can be written as a product of two integer numbers, 0 and 1 excluded.

This statement is a basic step in proving the Fundamental Theorem of Arithmetic (a.k.a Factorization Theorem). Hence Factorization Theorem can not be used to prove the statement.

Problem 2.(3pts)

Let \( m \) and \( r \) be two positive integers, and \( p \) a prime. Under which condition on \( m \) and \( r \) \( \sqrt[p]{m} \) is an irrational number. Prove it.

Problem 3.(2pts)

Two statements are given:

**Statement A:** Let \( a \) and \( b \) be two integers and \( p \) a prime. Then

\[
p|ab \implies p|a \lor p|b.
\]

**Statement B:** Let \( m \) and \( r \) be two positive integers and \( p \) a prime. Then

\[
p|m^r \implies p|m.
\]

Prove that Statement A implies Statement B.

2 Group B

Problem 1.(3pts)

Let \( x \) and \( y \) be two integers. Prove that

\[
3|(x - y) \implies 3|(x - y)^3
\]
Problem 2. (2pts)
Let \( r \) be an integer greater than 2. Prove that \( \sqrt{5} \) is an irrational number.

Problem 3. (3pts)
Let \( m \) be an integer. Prove that \( 3|m \) iff the sum of its digits is divisible by 3.

3 Group C

Problem 1. (2pts)
Let \( x \) and \( y \) be two integers. Prove that
\[
3|(x - y) \implies 3|(x - y)^2
\]

Problem 2. (2pts)
Write the contrapositive of the following statements.
(a) If \( 3|m \) \( \land \) \( 2|m \) then \( 6|m \).
(b) If a sequence is convergent then it is bounded.

Problem 3. (2pts)
Let \( m \) be an integer. Prove
\[
3|m^3 \implies 3|m
\]

Problem 4. (2pts)
Let \( m \) be an integer smaller than 10000. Prove that \( 9|m \) iff the sum of its digits is divisible by 9.