introduction into Mathematical Systems
Math3390
Spring 2010
Math Induction and Indentities

Group A

1 Problem - 5ex
Let \( p \in \mathbb{N} \), such that \( p \geq 2 \). Prove by math induction that:

\[
(p - 1) \mid p^n - 1 \quad \forall n \in \mathbb{N}
\]

2 Problem - 5ex
Let \( A \) be a set. Prove using math induction that

\[
|A| = n \Rightarrow |\mathcal{P}(A)| = 2^n
\]

i.e., that the cardinality of the power set of set \( A \) with \( n \) elements is \( 2^n \).

3 Problem - 5ex
Let \( F_n \) be a sequence of Fibonacci numbers, i.e.,

\[
F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}
\]

Prove by math induction that \( 2 | F_n \iff 3 | n \).

4 Problem - 5ex
Prove the identity

\[
\frac{n!}{n^n} = (1 - \frac{1}{n})(1 - \frac{2}{n})(1 - \frac{3}{n}) \cdots (1 - \frac{n-1}{n}) \quad \forall n \in \mathbb{N}
\]

5 Problem - 5ex
Find the formula for:

\[
1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!
\]

Then prove it using the math induction.
6 Problem - 5ex
Prove or disprove:
Let \( p^2 + q^2 = 1 \). Then
\[
p^2(p^2 - 1) = q^2(q^2 - 1)
\]

Group B

7 Problem - 5ex
Prove using math induction that
\[
16|5^n - 4n - 1
\]

8 Problem - 5ex
Prove using math induction that
\[
1 + 5 + 9 + \cdots + (4n - 3) = 2n^2 - n \quad \forall n \in \mathbb{Z}
\]

9 Problem - 5ex
A sequence \( a_n \) is defined recursively by
\[
a_1 = 1, \quad a_2 = 4, \quad a_3 = 9, \quad a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n - 3) \quad \text{for} \quad n \geq 4.
\]
Conjecture a formula for \( a_n \) and verify by math induction that your conjecture is correct.

10 Problem - 5ex
Prove the identity
\[
x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)
\]

11 Problem - 5ex
Prove the identity
\[
e^{-\lambda_1}e^{-\lambda_2}\cdots e^{-\lambda_n} = e^{-2\lambda}
\]
providing \( t_1 + t_2 + \cdots + t_n = 2 \).

12 Problem - 5ex
Prove the identity
\[
\left(\frac{2st}{t^2 + s^2}\right)^2 + \left(\frac{t^2 - s^2}{t^2 + s^2}\right)^2 = 1
\]
Group C

13 Problem - 5ex
Prove using math induction that
\[ 2^n \geq n^3 \quad \forall n \geq 10. \]

14 Problem - 5ex
Prove using math induction that
\[ 7|8^n - 1 \]

15 Problem - 5ex
Prove using math induction that
\[ 7|3^{2n} - 2^n \]

16 Problem - 5ex
A sequence \( a_n \) is defined recursively by
\[ a_1 = 1, \quad a_2 = 2, \quad a_n = a_{n-1} + 2a_{n-2} \quad \text{for} \quad n > 2. \]
Conjecture a formula for \( a_n \) and verify by math induction that your conjecture is correct.

17 Problem - 5ex
Prove the identity
\[ x^2 - y^2 = (x + y)(x - y) \]

18 Problem - 5ex
Prove the identity
\[ x^3 - y^3 = (x - y)(x^2 + xy + y^2) \]

19 Problem - 5ex
Prove or disprove:
Let \( p + q = 1 \). Then
\[ p^3 + q^3 = 1 - 3pq \]