From a Puzzle to a Theory – Creating a Mathematical Theory

Starting Puzzle. Among 8 golden rings one is a fake. The fake ring is slightly lighter than the others. Using a balance scale find the fake ring in only two measurements.

Puzzle B. Among 9 golden rings one is a fake. The fake ring is slightly lighter than the others. Using a balance scale find the fake ring in only two measurements.

Puzzle C. Among 10 golden rings one is a fake. The fake ring is slightly lighter than the others. Using a balance scale find the fake ring in only two measurements.

This puzzle cannot be solved. So we have to change, invert the question.

Puzzle D. Among 10 golden rings one is a fake. The fake ring is slightly lighter than the others. Use a balance scale to find the fake ring. What is the minimal number of the measurements needed to find the fake ring?

Answer: 3

Remark: To answer the question you need to do two things. Find the algorithm which would find the fake ring and then you have to prove it is the best possible.
In order to prove it we used the Worst Case Scenario(WCS).

Puzzle E. Among 11(or 12 or 13 or 14 or 15) golden rings one is a fake. The fake ring is slightly lighter than the others. Use a balance scale to find the fake ring. What is the minimal number of the measurements needed to find the fake ring?

Answer: 3 for all of them.

We can make a table.

<table>
<thead>
<tr>
<th># of rings</th>
<th># of measurements</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
</tr>
</tbody>
</table>
Time to change the question.

**Puzzle F.** Among \( N \) golden rings one is a fake. The fake ring is slightly lighter than the others. Use a balance scale to find the fake ring. What is the **minimal number** of the rings needed so that the number of measurements needed to find the fake ring is 4?

*Answer:* 28.

**Remark.** In order to show it we used the **Pigeonhole Principle.** Here is how. We need to split 28 rings in three groups. By the Pigeonhole principle at least one group must have 10 rings. So we split 28 = 9 + 9 + 10. By the WCS the fake ring could end up in the group of 10. For the 10 rings we know from before that the number of measurements needed is 3. This makes it 4 altogether.

**Puzzle G.** Among \( N \) golden rings one is a fake. The fake ring is slightly lighter than the others. Use a balance scale to find the fake ring. What is the **minimal number** of the rings needed so that the number of measurements needed to find the fake ring is 5 (or 6, or 7)?

*Answer:* for 5 we need 82 rings, for 6 we need 244 rings, ...

Form a table.

<table>
<thead>
<tr>
<th># of measurements</th>
<th># of rings</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>28</td>
</tr>
<tr>
<td>5</td>
<td>82</td>
</tr>
<tr>
<td>6</td>
<td>244</td>
</tr>
</tbody>
</table>
**Question G.** Find the expression for the function $F$, 

$$F(\text{# of measurements}) = \# \text{of rings}$$

**Daniel K. Conjecture 1.** $F(x) = 2 \cdot 3^{x-1} + 2 \cdot 3^{x-2} + 2 \cdot 3^{x-3} + \ldots + 2 \cdot 3^1$

This cannot be true since $F(3)=10$, but Daniel K.’s formula yields  

$$F(3) = 2 \cdot 3^{3-1} + 2 \cdot 3^{3-2} = 24$$

We disproved the Daniel K’s Conjecture 1. We used an example which proved the identity is not correct. This is called a **counterexample**.

**Daniel K. Conjecture 2.** $F(x) = 3^{x-1} + 1$

We first checked the formula for the numbers in the table. Then we tried to proved it. In order to prove it we needed the math induction.

So what is next for our little theory.  
We can go back and answer the generalized version of the Puzzle D.

**Puzzle DG(D generalized).** Among $N$ golden rings one is a fake. The fake ring is slightly lighter than the others. Use a balance scale to find the fake ring. What is the **minimal number** of the measurements needed to find the fake ring? More precisely find the expression for the function $G$, 

$$G(\text{# of rings}) = \# \text{of measurements}.$$ 

The next step in our theory was proposed by Michael N.

**Question A2.** Among 10 golden rings two rings are fake. The fake rings are slightly lighter than the others. Using a balance scale find the fake rings. What is the **minimal number** of the measurements needed to find the fake rings?

**Competition !!!** The best answer will receive 5 ex, the next best 4ex, then 3 and so on.