Instructions: This is an in-class part of Final Test. The take-home part will be announced on the course web site no later then Thursday afternoon. For take-home part students will be split into three groups according to their scores on the in-class part of the test. Students who score 18 or higher on the in-class part of the test will be in group A, students who score 10-17 will be in group B and students who score 9 or less in group C. The groups and scores will be listed on the course web page. Deadline to submit take-home part is Monday, May 4, 9:30am.

Problem 1 (3pts) Let $\mathfrak{P}$ be a set of all polynomials with degree 2 or less. Let $\Omega = \{ p \in \mathfrak{P} \mid p(0)=p(1) \}$. Let $T = \{ q(x) = ax^2 + bx + c \mid a+b=0 \}$. Show that $\Omega = T$.

Problem 2. (3pts) Write a generalization of the following statement:
Statement: Let $a$ and $b$ be two non-negative real numbers. Then
$$\frac{a+b}{2} \geq \sqrt{ab}$$

Problem 3. (3 pts) The following is the proof of the statement from the previous problem:

Proof:

$$\frac{(a-b)^2}{4} \geq 0$$
$$\Rightarrow a^2 - 2ab + b^2 \geq 0$$
$$\Rightarrow a^2 + 2ab + b^2 \geq 4ab$$
$$\Rightarrow \frac{(a+b)^2}{4} \geq ab$$
$$\Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$$

Is the proof correct? If it is correct, which method of proving has been used? If it is incorrect, indicate the steps which have been done incorrectly and explain why!

Problem 4. (3pts) The following statement is given:

*Any integer number is divisible by 2 or is larger than 100.*

(a) Write the statement using logical operands and quantifiers.

(b) Negate the statement and write it in plain English without logical operands and quantifiers.
Problem 5. (3pts) Is it possible to prove the following statement by mathematical induction?  
Statement:  
$$2^n3^n = 6^n \quad \forall n \in \mathbb{Z}.$$  
If it is possible, prove it using math inductions, if it is not possible, explain why not!

Problem 6. (3pts) Statement: Let \( p \) and \( q \) be two non-negative numbers such that \( p + q = 1 \). Then  
\[ p^2 + q^2 = 1 - 2pq. \]  
Two proofs of the statement are given. Which one is correct and which one is not? Explain.  
Proof A.:  
\[
\begin{align*}
   p + q &= 1 \\
   \Rightarrow (p + q)^2 &= 1^2 \\
   \Rightarrow p^2 + 2pq + q^2 &= 1 \\
   \Rightarrow p^2 + q^2 &= 1 - 2pq
\end{align*}
\]

Proof B.:  
\[
\begin{align*}
   p^2 + q^2 &= 1 - 2pq \\
   \Rightarrow p^2 + 2pq + q^2 &= 1 \\
   \Rightarrow (p + q)^2 &= 1 \\
   \Rightarrow 1^2 &= 1
\end{align*}
\]

Problem 7. (3pts) The formal system GEO is defined as follows:  
The GEO system uses three types of symbols (objects):  
\( \triangle, \square, \bigcirc \)  
The axioms of GEO are:  
Axiom 1.: \( \triangle \triangle \) is a theorem.  
Axiom 2.: If \( X \) is a theorem then \( X \square \) and \( X \bigcirc \) are theorems as well.  
Axiom 3.: If \( \square \square \) or \( \bigcirc \bigcirc \) is a part of a theorem then they can be replaced by \( \triangle \).  
(a) Create a theorem in the formal system GEO.  
(b) Which one of two statements below is a theorem in GEO? Explain why one is and the other is not a theorem in GEO! Prove the one which is a theorem.  
Statement 1.: \( \triangle \bigcirc \bigcirc \square \triangle \)  
Statement 2.: \( \triangle \square \triangle \bigcirc \bigcirc \)  

Problem 8. (3pts) Let \( P \) and \( Q \) be two equivalence relations on a set \( \Omega \). Let \( R = P \cup Q \). Is \( R \) an equivalence relation on \( \Omega \)?

Problem 9. (1pts) Determine whether the statement below is true or false.  
\textit{This sentence is false.}