Introduction into Mathematical Systems - Spring 2009
Takehome Test 1

Name: ____________________________
Nickname: _________________________

Instructions: Check your group assignment on the Message Board. The deadline to submit the take-home part is Wednesday, July 8, before the regular class.

Group A

Problem 1. (6pts) Given are 5 points on the plane. We are connecting these points with line segments. The line segments we call edges. Answer the following questions.
(a) What is the maximal number of edges we can draw? Explain (Prove it.).
(b) What is the minimal number of edges to guarantee that at least one point is connected to at least two points? Explain (Prove it.).
(c) What is the minimal number of edges needed to guarantee that you have a triangle? Explain (Prove it.).
(d) Repeat (a)-(c) in case of N points.

Problem 2. (6pts) A number is called an algebraic number if it is a root of a polynomial with the integer coefficients. For example, $\sqrt{2}$ is an algebraic number because it is a root of the polynomial $p(x) = x^2 - 2$. Recall to be a root it means $p(\sqrt{2}) = 0$.
(a) Show that $\sqrt{2} + \sqrt{3}$ is an algebraic number, i.e., find the polynomial with integer coefficients such that $\sqrt{2} + \sqrt{3}$ is its root. Hint: Calculate $(\sqrt{2} + \sqrt{3})^2$, $(\sqrt{2} + \sqrt{3})^3$, $(\sqrt{2} + \sqrt{3})^4$.
(b) Given are two primes $p$ and $r$. Show that $\sqrt{p} + \sqrt{r}$ is an algebraic number, i.e., find the polynomial with integer coefficients such that $\sqrt{p} + \sqrt{r}$ is its root.

Group B

Problem 1. (4pts)
(a) Prove or disprove: In a group of 28 people there are at least 3 of them born in the same month.
(b) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. If 5 integers are chosen from $A$, must at least one pair of the integers have a sum of 9?
(c) Prove a generalization of (b). Let $A = \{1, 2, 3, \ldots, 2k\}$ where $k$ is a positive integer. If $k + 1$ integers are chosen from $A$, must at least one pair of the integers have a sum of $2k + 1$?

Problem 2. (4pts) Prove the following statement by contraposition.
Let $m^4$ be divisible by 5 then $m$ is divisible by 5.
Be sure to give the complete proof.

Problem 3. (4pts) Two statements are given below. Only one is true, which one? Prove the true statement and disprove the incorrect statement.
Let $r > 0$ then \[
\frac{2r}{1 + r^2} \leq 1.
\]
Let $r > 0$ then \[
\frac{2r}{1 + r^2} > 1.
\]

**Group C**

**Problem 1** (2pts) Let $A, B$ be two sets. Prove or disprove

\[(A \setminus B)^c \cap (B \setminus A)^c = (A \cup B)^c \cup (A \cap B)\]

**Problem 2** (2pts) Construct a truth table for the logical expression

\[(P \land (P \implies Q)) \iff Q\]

**Problem 3.** (2pts) The statement is given:

If $n$ is an even number then $n^2$ and $n^3$ are integers divisible by 8.

(a) Write the contrapositive of the statement.
(b) Write the negation of the statement.

**Problem 4.** (2pts) Prove the following statement by contraposition.

Let $5 | n^2$ (this means $n^2$ is divisible by 5) then $5 | n$.

**Problem 5.** (2pts) Prove the following statement by contradiction.

Let $0 < r < 1$ then $3r^2 + 1 < 5 + r$.

**Problem 6.** (2pts) We form a sequence of numbers in the following way:
The first in the sequence is 7, the second and every other one is obtained by multiplying the previous one by 17 and dropping everything but the last digit.
So the second one is 9, since $7 \times 17 = 119$ and the last digit is 9. Now you multiply $9 \times 17 = 153$ so the third element of the sequence is 3. The question is what is the 353-rd element of the sequence?