Introduction into Mathematical Systems -
Spring 2009
Takehome Test 2

Name: ____________________________
Nickname: ____________________________

Instructions: Check your group assignment on the Message Board. The deadline to submit the take-home part is Wednesday, March 4, before the regular class.

Group A

Problem 1.(6pts) Using the strong principle of mathematics induction prove the following statement.

Let \( F_0 = 0 \) and \( F_1 = 1 \). and furthermore let

\[
F_{n+1} = F_n + F_{n-1} \quad \forall n \in \mathbb{N}.
\]

Show that

\[
F_n = \frac{1}{\sqrt{5}} (\phi^n - \psi^n)
\]

where

\[
\phi + \tau = 1 \quad \phi^2 = 1 + \phi \quad \psi^2 = \psi + 1.
\]

Further \( \phi > 0 \) and \( \psi < 0 \).

Problem 2.(6pts) Prove that any integer \( N \) larger than 5 can be written as

\[
N = 2k + 3l \quad \text{where} \ k, \ l \in \mathbb{Z}.
\]

Group B

Problem 1(4pts) Prove that \( 6|n^3 + 3n^2 + 2n \) for all positive integers \( n \).

Problem 2.(4pts) Prove by math induction:

\[
1^3 + 2^3 + 3^3 + \ldots + n^3 = \left( \frac{n(n+1)}{2} \right)^2
\]

Problem 3.(4pts) Find the least \( n \) for which the statement is true and then prove that \( (1 + n^2) < 2^n \)

Group C
Problem 1. Let $a_0 = 5$. Let
\[ a_{n+1} = 7a_n \quad n = 1, 2, \ldots \]
Prove $a_n = 5 \cdot 7^n$.

Problem 2. Prove that $24 | (5^{2n} - 1)$.

Problem 3. Let $a_1 = 0$ and $a_2 = 7$. Let
\[ a_{n+1} = a_{n-1} + 6n^2 + 2 \]
Find the formula for $a_n$ and then prove it using math induction.

Problem 4. Prove that
\[ (x + y)^4 - (x - y)^4 = 8xy \]
providing $x^2 + y^2 = 1$. 