A Subsidized Vickrey Auction for Cost Sharing

Jesse A. Schwartz†
Kennesaw State University

Quan Wen‡
Vanderbilt University

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Abstract

It is well-known that there is no cost-sharing mechanism that is budget balanced, efficient, and dominant strategy incentive compatible (DSIC). The Vickrey auction is DSIC and efficient, but raises surplus revenue. In an environment where players have constant marginal values, we introduce the subsidized Vickrey auction (SVA) that uses surplus revenue to offset some of the production costs. By compromising efficiency, the SVA improves the players’ payoffs over the Vickrey auction. We show that the SVA is DSIC, budget balanced, and value effective (awarding quantity only to the player who values it most), and Pareto optimal among all mechanisms with these three properties. We demonstrate that there is no welfare ranking between the SVA and the non-value effective serial cost-sharing mechanism.

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†Department of Economics, Finance, and Quantitative Analysis, Kennesaw State University, 1000 Chastain Road, Box 0403, Kennesaw, GA 30144, U.S.A. Email: jschwar7@kennesaw.edu

‡Department of Economics, Vanderbilt University, VU Station B #351819, 2301 Vanderbilt Place, Nashville, TN 37235-1819, U.S.A. Email: quan.wen@vanderbilt.edu
1 Introduction

In cost-sharing problems, a group of players must decide how much each consumes and pays for a jointly produced private good. Examples include farmers sharing an irrigation system and divisions of a corporation using the same advertising department or training facility. A cost-sharing mechanism specifies the allocation based on players’ reports of their private information. In this setting, it is generally impossible for a mechanism to be simultaneously budget-balanced, dominant strategy incentive compatible (DSIC), and allocatively efficient. For example, the Vickrey (1961) auction is DSIC and allocatively efficient but raises surplus revenue. The players have the thorny issue of what to do with this revenue: redistributing it among the players could upset DSIC, destroying it would be wasteful.

In this paper, we introduce a cost-sharing mechanism, called the subsidized Vickrey auction (SVA), that uses the revenue generated in a Vickrey-like auction to subsidize production. In a model with constant marginal values, we show that our SVA is DSIC and budget-balanced, but not allocatively efficient. For all DSIC direct mechanisms, the taxation principle of Rochet (1985) and Guesnerie (1995) establish that each player must face a price schedule that is independent from her own reported information. We exploit this characterization of DSIC mechanisms to show that the Vickrey auction is Pareto dominated by the SVA, where every player faces a lower price schedule. Furthermore, we prove that the SVA is Pareto optimal among all value-effective mechanisms. A mechanism is value effective if for whatever quantity is produced, it is allocated among the players to maximize their aggregate utility, even if the total quantity does not maximize social welfare. Although this notion of value effectiveness has not been used in the cost sharing literature, we adapt it from the environmental economics literature: pollution control is said to be cost effective if the given amount of pollution reduction is achieved at least cost, even if the total amount of pollution reduction does not maximize social welfare. Both value and cost effectiveness are used to relax full efficiency when efficiency is unattainable. In environmental economics,
the social planner may not know the social benefit of reducing pollution. In cost sharing, it is impossible to achieve full efficiency, DSIC and budget-balanced (as we show in Section 2, where the Vickrey auction is DSIC and efficient, but raises surplus revenue). Although value effectiveness eliminates any incentive for players to trade among themselves upon completion of the mechanism, it is sometimes possible to improve players’ expected payoffs by further sacrificing value effectiveness. The well-known serial cost-sharing mechanism of Moulin and Shenker (1992) is DSIC and budget-balanced, but it is generally not value effective.\footnote{See also Moulin and Shenker (1994) and Friedman and Moulin (1999) for more on the serial cost-sharing mechanism.} Nevertheless, we show that there is no Pareto ranking between the serial cost-sharing mechanism and SVA.

Our paper fits into the emerging literature on trading off allocative efficiency and budget-balancedness among DSIC mechanisms. An early example is McAfee’s (1992) double auction that sometimes sacrifices the least efficient trade to set the price for the other traders, where the loss of efficiency becomes negligible as the number of traders grows. More recently in rationing problems, Moulin (2009) considers how to design the lump-sum payments in an allocatively efficient mechanism to alleviate budget imbalance, while de Clippel, Naroditskiy and Greenwald (2009) further allow for allocative inefficiency. For cost-sharing problems, Moulin and Shenker (2001), Roughgarden and Sundararajan (2006), and Mehta, Roughgarden and Sundararajan (2009) strengthen DSIC to coalition strategy-proofness, allowing for either budget imbalance, or inefficiency, or both. Other than Mehta, Roughgarden and Sundararajan (2009), these papers deal exclusively with binary problems—each player is allocated either zero or one unit. Many of these mechanisms generate surplus revenue that goes unclaimed by any of the players. Alternatively, our SVA uses generated revenue to subsidize production beyond the efficient quantity, while maintaining DSIC and budget-balancedness in a cost-sharing environment with continuous quantities. Like these papers, our results are limited to players with linear demand. On the other hand, the serial cost-sharing mecha-
nism of Moulin and Shenker (1992) is specified much more generally, allowing for players with convex preferences, including quasi-linear demands. There are different ways to extend our SVA to allow for quasi-linear demands. In an earlier version of this paper (Schwartz and Wen, 2007), we had pursued one extension that still Pareto dominated the Vickrey auction and generated enough revenue to cover the costs, but it no longer allocated value effectively or balanced the budget.

In the next section, we specify the cost-sharing problem with linear demands and apply the Vickrey auction as a benchmark. In Section 3, we first introduce the SVA and establish its optimality among all DSIC, budget-balanced, and value-effective mechanisms. We then demonstrate with an example that there is no Pareto ranking between the SVA and the serial cost mechanism. Section 4 contains concluding remarks.

2 Model and Vickrey Auction

A set of \( n \geq 2 \) players, denoted as \( N \equiv \{1, \ldots, n\} \), jointly produce a private good and decide how much each consumes and pays. An outcome \( (q, t) = (q_1, \ldots, q_n, t_1, \ldots, t_n) \in R_+^n \times R^n \) specifies each player \( i \)'s quantity \( q_i \) and payment \( t_i \). The cost of producing \( q_1 + \cdots + q_n \) units is \( C(q_1 + \cdots + q_n) \). We assume that the cost function \( C(\cdot) \) is strictly convex, strictly increasing, and differentiable with marginal cost \( c(\cdot) \equiv C'(\cdot) \). As common in the cost-sharing literature, we assume that \( C(0) = 0 \), so that only variable cost is considered.

Let \( \theta = (\theta_i, \theta_{-i}) \) be the profile of players’ values, where \( \theta_i \) is player \( i \)'s value and \( \theta_{-i} \) is the profile of the other players’ values. Assume that \( \theta \in [0,1]^n \) is continuously distributed with full support. Each player \( i \) has linear utility \( u_i(q_i, t_i, \theta_i) = \theta_i q_i - t_i \) that depends only on her private value \( \theta_i \) and her part of the outcome \( (q_i, t_i) \).

Outcome \((q, t)\) is feasible if \( t_1 + \cdots + t_n \geq C(q_1 + \cdots + q_n) \), and budget-balanced if \( t_1 + \cdots + t_n = C(q_1 + \cdots + q_n) \). Outcome \((q, t)\) Pareto dominates outcome \((q', t')\) if \( u_i(q_i, t_i, \theta_i) \geq u_i(q_i', t_i', \theta_i) \) for all \( i \in N \), and \( u_i(q_i, t_i, \theta_i) > u_i(q_i', t_i', \theta_i) \) for at least one player \( i \in N \).
A feasible outcome is *Pareto optimal* if it is not Pareto dominated by any other feasible outcome. Outcome \((q, t)\) is *allocatively efficient* if \(q\) maximizes the total welfare:

\[
q \in \arg\max_{q' \in R^n} \sum_{i \in N} \theta_i q_i' - C \left( \sum_{i \in N} q_i' \right).
\]

Observe that Pareto optimality depends on both \(\theta\) and \(t\), but allocative efficiency depends only on \(\theta\). Outcome \((\theta, t)\) is *value effective* if the total quantity cannot be reassigned to improve the total welfare. With linear demands, \((q, t)\) is value effective if and only if \(\sum_{\hat{\psi} \in \hat{\psi}} \leq \max_{\psi \in \psi} \theta_i \) implies \(q_i = 0\). An allocative efficient outcome must be value effective, but not *vice versa*, because the total output in a value effective outcome may not be optimal.

In a *direct cost-sharing mechanism*, each player \(i \in N\) reports her value from \([0, 1]\). For each value profile \(\theta \in [0, 1]^n\), a quantity rule and payment rule of the mechanism specify an outcome \((q(\theta), t(\theta)) \in R_+^n \times R^n\). A mechanism is feasible (budget-balanced) if \((q(\theta), t(\theta))\) is feasible (budget-balanced) for all \(\theta \in [0, 1]^n\). A direct mechanism is *dominant strategy incentive compatible* (DSIC) if for all \(i \in N\) and for all \(\theta_i, \hat{\theta}_i, \text{ and } \theta_{-i}\):

\[
u_i(q_t(\theta_i, \theta_{-i}), t_i(\theta_i, \theta_{-i}), \theta_i) \geq u_i(q_t(\hat{\theta}_i, \theta_{-i}), t_i(\hat{\theta}_i, \theta_{-i}), \theta_i),
\]

so that it is always optimal for a player to truthfully report her value.

The Vickrey (1961) auction is a direct mechanism in which each player \(i \in N\) reports her value \(\hat{\theta}_i \in [0, 1]\), quantity is awarded to guarantee allocative efficiency presuming the players report truthfully, and payments are designed to ensure DSIC. Specific to our model,

\[
q^V_i(\hat{\theta}) = \begin{cases} 
  c^{-1}(\hat{\theta}_i) & \text{if } \hat{\theta}_i > \max_{j \neq i} \hat{\theta}_j \\
  c^{-1}(\hat{\theta}_i)/m & \text{if } \hat{\theta}_i = \max_{j \neq i} \hat{\theta}_j \\
  0 & \text{otherwise}
\end{cases}
\]

and

\[
t^V_i(\hat{\theta}) = \int_0^{q^V_i(\hat{\theta})} \max_{j \neq i} \{\hat{\theta}_j, c(z)\} dz,
\]

where in the case of ties, \(m\) is the total number of players with the highest report. In other words, the Vickrey auction assigns all production to the player with the highest report up to where the marginal cost is equal to her reported value. This player pays the opportunity
cost for this quantity, made up by the value \( \max_{j \neq i} \hat{\theta}_j \) for the quantity she displaces and the cost of additional quantity produced. Figure 1 below illustrates the winning player \( i \)'s quantity and payment \((q_i^V(\hat{\theta}), t_i^V(\hat{\theta}))\) in the Vickrey auction.

\[
\begin{align*}
\hat{\theta}_i & \quad \text{\( c(\cdot) \)} \\
0 & \quad \hat{\theta}_i \\
\max_{j \neq i} \hat{\theta}_j & \quad \text{\( \hat{\theta}_i \)} \\
q & \quad \text{\( c(\cdot) \)} \\
q_i^V & \quad \text{\( \hat{\theta}_i \)} \\
0 & \quad \hat{\theta}_i
\end{align*}
\]

\textbf{Figure 1:} Winning player \( i \)'s allocation in the Vickrey auction

Observe that (i) each player \( i \) faces a price schedule \( \max_{j \neq i} \{\hat{\theta}_j, c(z)\} \) that is independent of her own report and (ii) player \( i \) wins all of the quantity that is priced below her reported value. It is these two features that make it always optimal for a player to report her true value, establishing the Vickrey auction as a DSIC mechanism. Schwartz and Wen (2008) formulate \textit{perfect price discriminating} (PPD) mechanisms with these two features to characterize any DSIC mechanism regardless of whether the mechanism is efficient or not. A PPD mechanism is a direct mechanism where player \( i \)'s quantity and payment are

\[
q_i(\hat{\theta}_i, \hat{\theta}_{-i}) \in \arg \max_{q_i} \int_0^{q_i} \left[ \hat{\theta}_i - p_i(z, \hat{\theta}_{-i}) \right] dz, \tag{1}
\]

\[
t_i(\hat{\theta}_i, \hat{\theta}_{-i}) = \int_0^{q_i(\hat{\theta}_i, \hat{\theta}_{-i})} p_i(z, \hat{\theta}_{-i}) dz + L_i(\hat{\theta}_{-i}), \tag{2}
\]

where \( p_i(\cdot, \hat{\theta}_{-i}) \) is the price schedule player \( i \) faces and \( L_i(\hat{\theta}_{-i}) \) is her lump-sum payment.\(^2\)

Neither \( p_i(\cdot, \hat{\theta}_{-i}) \) nor \( L_i(\hat{\theta}_{-i}) \) depends on player \( i \)'s own report. The price schedule \( p_i(\cdot, \hat{\theta}_{-i}) \)

\(^2\)The specification in (1) applies to the model with a linear utility function; see Schwartz and Wen (2008) for the case with quasilinear utility.
determines how much player $i$ would pay for any quantity she wins. The quantity player $i$ does win by (1) maximizes her payoff given this price schedule. Schwartz and Wen (2008) show that any direct mechanism is DSIC if and only if it is a PPD mechanism. This result allows us to focus on price schedules in constructing new DSIC mechanisms.

To illustrate the failings of the Vickrey auction for cost-sharing, consider an example with linear marginal cost $c(q) = q$ and with $n = 2$ players whose values are independently and uniformly distributed. Player $i$’s *ex ante* payoff in the Vickrey auction is

$$
\int_0^1 \int_0^{\theta_i} \left( \theta_i^2 - \theta_j^2 - \int_{\theta_j}^{\theta_i} q\, dq \right) \, d\theta_j \, d\theta_i = \frac{1}{12}.
$$

Observe that the winner’s payment exceeds the production cost, and this overpayment is not accounted for in the players’ welfare. In this truth-telling equilibrium, the expected overpayment made by player $i$ is

$$
\int_0^1 \int_0^{\theta_i} \frac{1}{2} \theta_j^2 \, d\theta_j \, d\theta_i = \frac{1}{24}
$$

which is 50% of player $i$’s *ex ante* payoff. In cost-sharing situations, players jointly produce the good themselves: there is no opposing seller to produce the good and pocket this overpayment. One possible option for the players is to sell the rights for any overpayment to an outside party, which induces a budget-balanced mechanism. However, this option may not be practical in reality, which may be the reason that the Vickrey auction has not been applied for cost-sharing problems.

The question is whether the overpayment in the Vickrey auction can be used to improve the players’ welfare without destroying their incentives to report truthfully. Simply refunding the overpayment to the winning player does not work because a losing player sometimes has an incentive to exaggerate her report. We next find a better use for the overpayment.

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3 The taxation principle of Rochet (1985) and Guesnerie (1995) establishes a similar characterization of DSIC mechanisms for allocating heterogenous goods.
3 The Subsidized Vickrey Auction

We now construct a PPD mechanism, called the subsidized Vickrey auction, that uses the overpayment from the Vickrey auction to subsidize production. In the Vickrey auction, if player $i$ reports the highest value, her overpayment is

$$
\int_{0}^{c^{-1}(\hat{\theta}_j)} \left[ \hat{\theta}_j - c(z) \right] dz,
$$

where $\hat{\theta}_j = \max_{k \neq i} \hat{\theta}_k$.

The idea of our subsidized Vickrey auction is to keep the unit price at $\hat{\theta}_j$ for quantity beyond $c^{-1}(\hat{\theta}_j)$, until the overpayment is exhausted at quantity $x_i$ such that

$$
\int_{0}^{c^{-1}(\hat{\theta}_j)} \left[ \hat{\theta}_j - c(z) \right] dz = \int_{c^{-1}(\hat{\theta}_j)}^{x_i} \left[ c(z) - \hat{\theta}_j \right] dz.
$$

The right side of (3) represents player $i$’s underpayment for quantities between $c^{-1}(\hat{\theta}_j)$ and $x_i$. Next, construct a price schedule for player $i$:

$$
p_i^*(q, \hat{\theta}_{-i}) = \begin{cases} 
\hat{\theta}_j & \text{if } q \leq x_i \\
c(q) & \text{otherwise,}
\end{cases}
\text{where } \hat{\theta}_j = \max_{k \neq i} \hat{\theta}_k.
$$

as illustrated in Figure 2 (where the 45-degree line represents the marginal cost $c(\cdot)$).

![Figure 2: Player i’s price schedule in the SVA](image)

The subsidized Vickrey auction (SVA) is the PPD mechanism, denoted by $(q^*(\cdot), t^*(\cdot))$, where player $i$’s quantity and payment are given by (1) and (2) using $p_i^*(\cdot, \hat{\theta}_{-i})$ as the price.
schedule and zero lump-sum payments, along with any tie-breaking rule that guarantees budget balancedness. For example, when player \( i \) ties for the highest report, set \( q_i^*(\hat{\theta}_i, \hat{\theta}_{-i}) = 0 \). In the truth-telling equilibrium, ties occur with probability zero, and so the tie-breaking rule will not matter. Figure 3 illustrates the two possible outcomes in the SVA when player \( i \)'s report is highest.

![Figure 3: Player \( i \)'s outcome in the SVA](image)

We next establish various properties of the SVA.

**Proposition 1** The subsidized Vickrey auction is DSIC and budget balanced. The outcome of the truth-telling equilibrium is value effective.

**Proof.** The SVA is a PPD mechanism and thus DSIC. Consider any profile \( \theta = (\theta_1, \ldots, \theta_n) \), and denote by \( i \) the player with the highest value. If \( \theta_i = \theta_j \equiv \max_{k \neq i} \theta_k \), the mechanism is budget balanced by definition of the SVA. If \( \theta_i > \theta_j \), there are two cases to consider: First, when \( q_i^*(\theta_i, \theta_{-i}) = x_i \) as shown in the left panel of Figure 3, player \( i \) pays

\[ t_i^*(\theta_i, \theta_{-i}) = x_i \theta_j = \int_0^{x_i} c(z) \, dz = C(q_i^*(\theta_i, \theta_{-i})). \]

Second, when \( q_i(\theta_i, \theta_{-i}) > x_i \) as shown in the right panel of Figure 3, player \( i \) pays

\[ t_i^*(\theta_i, \theta_{-i}) = x_i \theta_j + \int_{x_i}^{q_i(\theta_i, \theta_{-i})} c(z) \, dz = \int_0^{q_i(\theta_i, \theta_{-i})} c(z) \, dz = C(q_i^*(\theta_i, \theta_{-i})). \]
In either case, player \( p \) pays exactly the cost for the quantity she wins, and no other players win anything or pay anything. Hence, the SVA is budget balanced.

The properties of DSIC and budget balanceness hold for all report profiles. In the truth-telling equilibrium, \( \hat{\theta} = (\theta_1, \ldots, \theta_n) \), and so (1) guarantees the outcome is value effective because only the player with the highest value can win positive quantity. ■

The Vickrey auction is also a PPD mechanism where player \( i \in N \) has price schedule \( p_i^v(q, \theta_{-i}) = \max_{j \neq i} \{\hat{\theta}_j, c(z)\} \) and a zero lump-sum payment. The Vickrey auction is feasible, but not budget balanced because the winner pays more than the cost of production. Unlike the Vickrey auction, the SVA is not allocatively efficient. In the truth-telling equilibrium, when the winner’s value \( \theta_i < c(x_i) \), as in the left panel of Figure 3, player \( i \) inefficiently wins \( x_i \) instead of the efficient quantity \( c^{-1}(\theta_i) \). Although the SVA may award too much to the winning player, in the equilibrium, quantity is never awarded to anyone without the highest value. Nevertheless, the sacrifice of efficiency in the SVA is not for nothing.

**Proposition 2** For almost all value profiles, the outcome of the truth-telling equilibrium in the subsidized Vickrey auction Pareto dominates that in the Vickrey auction.

**Proof.** Consider the truth-telling equilibrium in both mechanisms. For almost all profiles, \( \theta_i > \theta_j \iff \max_{k \neq i} \theta_k > 0 \) for some \( i \in N \). In both mechanisms, every player other than \( i \) obtains a zero payoff. In the Vickrey auction, player \( i \) wins \( q_i^v = c^{-1}(\theta_i) > c^{-1}(\theta_j) \). In the SVA, player \( i \) wins \( q_i^s = \max\{x_i, q_i^v\} \) and obtains payoff

\[
\begin{align*}
\int_0^{q_i^s} [\theta_i - p_i^s(z, \theta_{-i})] \, dz &= \int_0^{q_i^v} [\theta_i - \theta_j] \, dz + \int_{q_i^v}^{q_i^s} [\theta_i - p_i^s(z, \theta_{-i})] \, dz \\
&\geq \int_0^{q_i^v} [\theta_i - \theta_j] \, dz \\
&= \int_0^{c^{-1}(\theta_j)} [\theta_i - \theta_j] \, dz + \int_{c^{-1}(\theta_j)}^{q_i^v} [\theta_i - \theta_j] \, dz \\
&> \int_0^{c^{-1}(\theta_j)} [\theta_i - \theta_j] \, dz + \int_{c^{-1}(\theta_j)}^{q_i^v} [\theta_i - c(z)] \, dz \\
&= \int_0^{q_i^v} [\theta_i - p_i^v(z, \theta_{-i})] \, dz,
\end{align*}
\]
which is player $i$’s payoff in the Vickrey auction, noting that the strict inequality results because $c(q) > \theta_j$ for all $q \in [c^{-1}(\theta_j), \theta^V_i]$. ■

The example considered in section 2 with quadratic costs and uniformly distributed values demonstrates that the Pareto improvement given by Proposition 2 can be quite large. Each player’s ex ante equilibrium payoff in the SVA in this example is

\[
\int_0^1 \left[ \int_0^{\theta_1/2} \left( \theta_1 \theta_1 - \frac{\theta_1^2}{2} \right) d\theta_2 + \int_{\theta_1/2}^{\theta_1} \left( \theta_1 (2\theta_2) - \frac{(2\theta_2)^2}{2} \right) d\theta_2 \right] d\theta_1 = \frac{5}{48} > \frac{1}{12},
\]

which is each player’s equilibrium payoff in the Vickrey auction. Recall from this example that the Vickrey auction generates 50% of a player’s ex ante payoff as a loss from destroying excess revenue; the SVA reclaims half of this loss. More generally, the SVA outperforms all other feasible and DSIC mechanisms that yield value effective outcomes.

**Proposition 3** For almost all value profiles, the outcome of the truth-telling equilibrium in the subsidized Vickrey auction Pareto dominates that in any other feasible and DSIC direct mechanism such that in the truth-telling equilibrium: i) the outcome is value effective and ii) any player with value 0 receives a payoff of 0.

We relegate the proof of Proposition 3 to the Appendix. Proposition 3 implies that the total welfare in the SVA is at least as large as that in any mechanism with the same properties. One immediate implication is the well-known non-existence of a mechanism that is DSIC, budget balanced, and allocatively efficient, noting that budget balancedness and allocative efficiency imply feasibility and value effectiveness. If such a mechanism existed, it would give more total welfare than the SVA when the two highest values are close together and the SVA is allocatively inefficient, as in the left panel of Figure 3.

Proposition 3 establishes the Pareto optimality of the SVA only within the class of DSIC mechanisms that are value effective. Although value effectiveness eliminates any incentive for trade among the players upon completion of the mechanism, it may be possible to sacrifice
this property to improve ex ante welfare. The prominent serial cost-sharing mechanism of Moulin and Shenker (1992) is DSIC but not value effective. We next demonstrate that there is no Pareto ranking between the SVA and the serial cost-sharing mechanism in the example from Section 2. In the serial cost-sharing mechanism, two batches are produced. The two players equally share the quantity and cost of the first batch, and the player with higher value receives the second batch and pays its incremental cost. As illustrated in Figure 4, the serial cost-sharing mechanism can be represented as a PPD mechanism with the following price schedule for player $i \in \{1, 2\}$:

$$p_{i}^{SC}(q, \theta_{j}) = \begin{cases} 
2q & \text{for } 0 \leq q \leq \frac{1}{2}\theta_{j} \\
q + \frac{1}{2}\theta_{j} & \text{for } q > \frac{1}{2}\theta_{j}.
\end{cases}$$

![Figure 4: Player $i$'s price schedule in the serial cost-sharing mechanism](image)

The allocation and payment rules in the serial cost-sharing mechanism are:

$$q_{i}^{SC} = \begin{cases} 
\frac{1}{2}\theta_{i} & \text{if } \theta_{i} \leq \theta_{j} \\
\theta_{i} - \frac{1}{2}\theta_{j} & \text{if } \theta_{i} > \theta_{j},
\end{cases}$$

$$t_{i}^{SC} = \begin{cases} 
\frac{1}{2}C(2q_{i}^{SC}) & \text{if } \theta_{i} \leq \theta_{j} \\
\frac{1}{2}C(2q_{j}^{SC}) + C(q_{i}^{SC} + q_{j}^{SC}) - C(2q_{j}^{SC}) & \text{if } \theta_{i} > \theta_{j}
\end{cases}$$

$$= \begin{cases} 
\frac{1}{4}\theta_{i}^{2} & \text{if } \theta_{i} \leq \theta_{j} \\
\frac{1}{2}\theta_{i}^{2} - \frac{1}{4}\theta_{j}^{2} & \text{if } \theta_{i} > \theta_{j}.
\end{cases}$$

See also Moulin and Shenker (1994) and Friedman and Moulin (1999) for more on the serial cost-sharing mechanism.
For all \( \theta_i \geq \theta_j \), denote the maximal potential surplus by
\[
\omega = \theta_i^2 - C(\theta_i) = \frac{1}{2} \theta_i^2.
\]
The total welfare in the serial cost-sharing mechanism,
\[
w^{SC}(\theta_i, \theta_j) = \bar{w} - \frac{1}{2} \theta_j [\theta_i - \theta_j] = \frac{1}{2} [\theta_i^2 - \theta_i \theta_j - \theta_j^2],
\]
falls short of \( \bar{w} \) because quantity \( \frac{1}{2} \theta_j \) is awarded to the low-valued player \( j \) rather than the high-valued player \( i \). On the other hand, the total welfare in the SVA,
\[
w^{SVA}(\theta_i, \theta_j) = \begin{cases} 
\bar{w} & \text{if } \theta_i \geq 2 \theta_j \\
\bar{w} - \frac{1}{2} [2 \theta_j - \theta_i]^2 & \text{if } \theta_i < 2 \theta_j 
\end{cases}
\]
\[
= \begin{cases} 
\frac{1}{2} \theta_i^2 & \text{if } \theta_i \geq 2 \theta_j \\
2 \theta_j [\theta_i - \theta_j] & \text{if } \theta_i < 2 \theta_j,
\end{cases}
\]
falls short of \( \bar{w} \) because of over-producing \( 2 \theta_j - \theta_i \) when the two players’ values are close together; i.e., when \( \theta_j \leq \theta_i < 2 \theta_j \). Comparing these two mechanisms, we find that
\[
w^{SC} < w^{SVA} \text{ if and only if } \theta_j < \frac{1}{2} \left( 5 - \sqrt{5} \right) \theta_i.
\]
Figure 5 demonstrates that these two mechanisms cannot be Pareto ranked. For distributions such that the two values drawn are likely to be close to each other, players’ ex ante welfare is higher in the serial cost-sharing mechanism, and vice versa.

\[
\begin{array}{c}
\theta_2 \\
\hline
0 & 1 \\
\end{array}
\]

\[
\begin{array}{c}
\theta_1 \\
\hline
0 & 1 \\
\end{array}
\]

\[
\text{Figure 5: Comparison of } w^{SC} \text{ and } w^{SVA}
\]

Indeed, when values are uniformly distributed, the total ex ante welfare in each mechanism is equal to \( \frac{5}{24} \). The uniform distribution is a special case of the symmetric triangular
distribution with density function
\[
f_\alpha(x) = \begin{cases} 
1 + \alpha - 4\alpha x & \text{if } x \leq \frac{1}{2} \\ 
1 - 3\alpha + 4x & \text{if } x > \frac{1}{2}, 
\end{cases}
\]
where \( \alpha \in [-1, 1] \). The density function is \( \wedge \)-shaped when \( \alpha > 0 \) and \( \vee \)-shaped when \( \alpha < 0 \).

Welfare levels are
\[
W^{SC} = 2 \int_0^1 \int_0^{\theta_i} w^{SC}(\theta_i, \theta_j)f_\alpha(\theta_i)f_\alpha(\theta_j)d\theta_jd\theta_i = \frac{5}{24} + \frac{1}{24}\alpha,
\]
\[
W^{SVA} = 2 \int_0^1 \int_0^{\theta_i} w^{SVA}(\theta_i, \theta_j)f_\alpha(\theta_i)f_\alpha(\theta_j)d\theta_jd\theta_i = \frac{5}{24} + \frac{17}{480}\alpha - \frac{7}{576}\alpha^2.
\]
Observe that \( W^{SVA} > W^{SC} \) for all \( \alpha \in (-18/35, 0) \) and \( W^{SVA} > W^{SC} \) for all \( \alpha \in [-1, 18/35) \cup (0, 1] \). This example demonstrates that under the quite ordinary circumstances of independent private values, either mechanism may do better.

4 Conclusion

In many games of incomplete information, it is generally impossible to simultaneously achieve budget balance, allocative efficiency, and truthful bidding as a dominant strategy. Thus, choosing a mechanism to solve an allocation problem involves tradeoffs. The Vickrey auction and Vickrey-Clarke-Groves mechanisms more generally achieve allocative efficiency and dominant strategies, but are often plagued by budget imbalance. In this paper, we have shown that the budget surplus need not rule out altogether the Vickrey auction from use as a cost-sharing mechanism. We have proposed an auction that redistributes the Vickrey surplus back to players, not in direct monetary rebates, but rather in subsidized production. Although this subsidized Vickrey auction sacrifices some allocative efficiency, we have shown that it Pareto dominates all DSIC mechanisms that are feasible and value effective (awarding quantity only to the player who values it most). We then demonstrated that there is no welfare ranking between our subsidized Vickrey auction and the serial cost-sharing mechanism which is not value effective.
Appendix

Proof of Proposition 3: The proof is divided into four steps.

Any DSIC direct mechanism is a PPD mechanism with a non-decreasing price schedule. Given any DSIC direct mechanism with quantity rule \( q(\theta) \) and payment rule \( t(\theta) \) such that any player with value 0 receives a payoff of 0, consider the PPD mechanism with price schedules \( p_i(q, \theta_{-i}) = \inf \left\{ \hat{\theta}_i(q_i(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i, \theta_{-i}) \geq q \right\} \) and zero lump-sum payments for all \( i \in N \). DSIC guarantees that \( q_i(\theta_i, \theta_{-i}) \) is nondecreasing in \( \theta_i \) (Theorem 7.2 of Fudenberg and Tirole, 1991), thus implying that \( p_i(q, \theta_{-i}) \) is nondecreasing in \( \theta_i \). Observe that \( p_i(q, \theta_{-i}) \leq \theta_i \) for all \( q \leq q_i(\theta_i, \theta_{-i}) \) and that \( p_i(q, \theta_{-i}) \geq \theta_i \) for all \( q > q_i(\theta_i, \theta_{-i}) \), so that \( q_i(\theta_i, \theta_{-i}) \) is a solution to (1). With the same quantity rule \( q(\theta) \) and players with value 0 earning zero payoffs, both the original DSIC mechanism and this PPD mechanism must also share the same payment rule \( t(\theta) \) by standard payoff equivalence results, as given in Krishna (2010) for example.

Player i’s price schedule is bounded from below by the highest value of her opponents. Without loss of generality, we may restrict attention to PPD mechanisms such that for all \( \theta_{-i} \), we have \( p_i(q, \theta_{-i}) \geq \theta_j = \max_{k \neq i} \theta_k \) for all \( q > 0 \). Otherwise, suppose that \( p_i(q’, \theta_{-i}) < \theta_j \) for some \( q’ > 0 \). By the monotonicity of the price schedule, \( p_i(q, \theta_{-i}) \leq \theta_j \) for all \( q \leq q’ \). Then (1) would dictate that \( q(\theta_i, \theta_{-i}) \geq q’ > 0 \) for any \( \theta_i \) such that \( p_i(q’, \theta_{-i}) < \theta_i < \theta_j \), violating value effectiveness. Without loss of generality, we can restrict attention to the case that \( p_i(q, \theta_{-i}) \geq \theta_j \) for all \( q \geq 0 \), noting that the specification at \( q = 0 \) does not change the outcome of the mechanism.

There are no lump-sum payments. Consider a player \( i \) with \( \theta_i = 0 \). Because \( p_i(z, \theta_{-i}) \geq 0 \) for all \( \theta_{-i} \), (1) and (2) imply that player \( i \)'s equilibrium payoff is

\[
\max_{q \geq 0} \int_0^q [0 - p_i(z, \theta_{-i})] \, dz - L_i(\theta_{-i}) = -L_i(\theta_{-i}).
\]

Because any player with the lowest possible value receives zero payoff, it must the case that \( L_i(\theta_{-i}) = 0 \).

Player i’s price schedule is no lower than that in the SVA. There are three cases to
consider. For $\theta_i \leq \theta_j$, player $i$ has a zero payoff under both SVA and this PPD mechanism.

For $\theta_j < \theta_i \leq c(x_i(\theta_{-i}))$, there are two subcases. If $q(\theta_i, \theta_{-i}) \leq x_i(\theta_{-i}) = q_i^*$, then

$$\theta_i q_i^*(\theta) - t_i^*(\theta) = \int_0^{q_i^*} [\theta_i - p_i^*(z, \theta_{-i})] \, dz$$

$$= \int_0^{x_i(\theta_{-i})} [\theta_i - \theta_j] \, dz$$

$$= \int_0^{q(\theta_i, \theta_{-i})} [\theta_i - \theta_j] \, dz + \int_{x_i(\theta_{-i})}^{q(\theta_i, \theta_{-i})} [\theta_i - \theta_j] \, dz$$

$$\geq \int_0^{q(\theta_i, \theta_{-i})} [\theta_i - \theta_j] \, dz$$

$$\geq \int_0^{q(\theta_i, \theta_{-i})} [\theta_i - p_i(z, \theta_{-i})] \, dz$$

$$= \theta_i q_i(\theta) - t_i(\theta)$$

where the last inequality is due to $p_i(q, \theta_{-i}) \geq \theta_j$ for all $q \geq 0$ as shown above. On the other hand, if $q(\theta_i, \theta_{-i}) > x_i(\theta_{-i}) = q_i^*$, then

$$\int_0^{q_i^*} [\theta_i - p_i^*(z, \theta_{-i})] \, dz = \int_0^{x_i(\theta_{-i})} [\theta_i - \theta_j] \, dz$$

$$= \int_0^{x_i(\theta_{-i})} [\theta_i - c(z)] \, dz$$

$$> \int_0^{x_i(\theta_{-i})} [\theta_i - c(z)] \, dz + \int_{x_i(\theta_{-i})}^{q(\theta_i, \theta_{-i})} [\theta_i - c(z)] \, dz$$

$$= \int_0^{q(\theta_i, \theta_{-i})} [\theta_i - c(z)] \, dz$$

$$\geq \int_0^{q(\theta_i, \theta_{-i})} [\theta_i - p_i(z, \theta_{-i})] \, dz,$$

where the first equality is by the definition of $x_i(\theta_{-i})$, the first inequality is due to $c(q) > \theta_i$ for all $q \geq x_i(\theta_{-i})$, and the last inequality is due to the feasibility of this PPD mechanism.

Lastly for $\theta_i > c(x_i(\theta_{-i}))$,

$$\int_0^{q_i^*} [\theta_i - p_i^*(z, \theta_{-i})] \, dz = \max_q \int_0^{q_i} [\theta_i - c(z)] \, dz$$

$$\geq \int_0^{q(\theta_i, \theta_{-i})} [\theta_i - c(z)] \, dz$$

$$\geq \int_0^{q(\theta_i, \theta_{-i})} [\theta_i - p_i(z, \theta_{-i})] \, dz.$$
where the last inequality is due to the feasibility of this PPD mechanism.

References


