3.8 Chapter Review

3.8.1 Concepts to Know

You should have an understanding of, and be able to explain the concepts listed below.

1. Boundary and interior points

2. Open and closed sets

3. Bounded sets

4. Function of several variables, their domain.

5. Level curves

6. Limit of a function of several variables

7. Limit along a path

8. Continuity of functions of several variables

9. Definition and meaning of partial derivatives

10. Computing partial derivatives

11. The chain rule for functions of several variables

12. Application of the chain rule to implicit differentiation

13. Plane tangent to a function \( z = f(x, y) \) at a given point.

14. Linear approximation

15. The differential

16. Directional derivative

17. The gradient vector

18. Applications of the gradient vector

19. Fermat’s theorem for functions of several variables

20. Local and global extrema for functions of several variables.
3.8. **CHAPTER REVIEW**

### 3.8.2 True-False Questions

Decide if each question below is true or false. In each case, you must justify your answer. If your answer is true, then either prove it, or give a justification (theorem, definition). If the answer is wrong, give a counter example, or explain what would make the answer true.

1. \( f_y(a, b) = \lim_{y \to b} \frac{f(a, y) - f(a, b)}{y - b} \)

2. There exists a function with continuous second order partial derivatives such that
   \[
   f_x(x, y) = x + y^2
   \]
   and
   \[
   f_y(x, y) = x - y^2
   \]

3. \( f_{xy} = \frac{\partial^2 f}{\partial x \partial y} \)

4. \( D_z f(x, y) = f_z(x, y, z) \)

5. If \( f(x, y) \to L \) along every straight line through \((a, b)\) then \( \lim_{(x, y) \to (a, b)} f(x, y) = L \).

6. If \( f \) has a local minimum at \((a, b)\) and \( f \) is differentiable at \((a, b)\) then \( \nabla f(a, b) = \vec{0} \).

7. If \( f \) is a function then \( \lim_{(x, y) \to (2, 5)} f(x, y) = f(2, 5) \).

8. If \( f(x, y) = \ln y \) then \( \nabla f(x, y) = \frac{1}{y} \).

9. If \((2, 1)\) is a critical point of \( f(x, y) \) and \( f_{xx}(2, 1) f_{yy}(2, 1) < |f_{xy}(2, 1)|^2 \)
   then \( f \) has a saddle point at \((2, 1)\).

10. If \( \vec{u} \) is any vector, then \( D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \frac{\vec{u}}{\|\vec{u}\|} \).

### 3.8.3 Practice Problems

Do the following problems:

1. Find the domain, range and identify the level curves of \( f(x, y) = 9x^2 + y^2 \).

2. Find the domain, range and identify the level curves of \( f(x, y) = \sqrt{x^2 - y} \).

3. Find the domain, range and identify the level surfaces of \( f(x, y, z) = x^2 + y^2 - z \).

4. Find \( \lim_{(x, y) \to (x, \ln 2)} e^y \cos x \).
5. Find \( \lim_{(x,y) \to (1,1)} \frac{x - y}{x^2 - y^2} \).

6. By considering different paths of approach, show \( \lim_{(x,y) \to (0,0)} \frac{y}{x^2 - y} \) does not exist.

7. By considering different paths of approach, show \( \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{xy} \) does not exist.

8. Find all the first order partials for \( f(r, \theta) = r \cos \theta + r \sin \theta \).

9. Find all the first order partials for \( f(R_1, R_2, R_3) = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \).

10. Find all the second order partials for \( f(x, y) = y + \frac{x}{y} \).

11. Find all the second order partials for \( f(x, y) = e^x + y \sin x \).

12. Find all the second order partials for \( f(x, y) = x + xy - 5x^3 + \ln (x^2 + 1) \).

13. Find \( \frac{dw}{dt} \) at \( t = 0 \) if \( w = \sin (xy + \pi) \), \( x = e^t \) and \( y = \ln (t + 1) \).

14. Find \( \frac{\partial w}{\partial r} \) and \( \frac{\partial w}{\partial s} \) when \( r = \pi \) and \( s = 0 \) if \( w = \sin (2x - y) \), \( x = r + \sin s \) and \( y = rs \).

15. Find the directions in which \( f(x, y) = \cos x \cos y \) increases and decreases most rapidly at \( P_0 = \left( \frac{\pi}{4}, \frac{\pi}{4} \right) \) and find the rate of increase in each direction. Finally, find the derivative of \( f \) at \( P_0 \) in the direction of \( \vec{u} = (3, 4) \).

16. What is the largest value the directional derivative of \( f(x, y, z) = xyz \) can have at the point \( (1, 1, 1) \)?

17. Find an equation of the tangent plane and parametric equations of the normal line to the level surface \( x^2 - y - 5z = 0 \) at the point \( P_0 = (2, -1, 1) \).

18. Find an equation of the plane tangent to the surface \( z = \ln (x^2 + y^2) \) at the point \( (0, 1, 0) \).

19. Find the linearization of \( f(x, y, z) = xy + 2yz - 3xz \) at the points \( (1, 0, 0) \) and \( (1, 1, 0) \).

20. Find the local extrema of \( f(x, y) = x^2 - xy + y^2 + 2x + 2y - 4 \).

21. Find the local extrema of \( f(x, y) = 2x^3 + 3xy + 2y^3 \).
3.8. Answers to True-False Questions

Decide if each question below is true or false. In each case, you must justify your answer. If your answer is true, then either prove it, or give a justification (theorem, definition). If the answer is wrong, give a counter example, or explain what would make the answer true.

1. \( f_y (a, b) = \lim_{y \to b} \frac{f(a,y) - f(a,b)}{y-b} \).
   
   True

2. There exists a function with continuous second order partials derivatives such that

   \( f_x (x, y) = x + y^2 \)

   and

   \( f_y (x, y) = x - y^2 \)

   False

3. \( f_{xy} = \frac{\partial^2 f}{\partial x \partial y} \)
   
   False

4. \( D_uf (x, y, z) = f_x (x, y, z) \)
   
   True

5. If \( f (x, y) \to L \) along every straight line through \((a, b)\) then \( \lim_{(x,y) \to (a,b)} f (x, y) = L \).
   
   False

6. If \( f \) has a local minimum at \((a, b)\) and \( f \) is differentiable at \((a, b)\) then \( \nabla f (a, b) = 0 \).
   
   True

7. If \( f \) is a function then \( \lim_{(x,y) \to (2,5)} f (x, y) = f (2, 5) \).
   
   False

8. If \( f (x, y) = \ln y \) then \( \nabla f (x, y) = \frac{1}{y} \).
   
   False

9. If \((2, 1)\) is a critical point of \( f (x, y) \) and \( f_{xx} (2, 1) f_{yy} (2, 1) < [f_{xy} (2, 1)]^2 \) then \( f \) has a saddle point at \((2, 1)\).

10. If \( \overrightarrow{u} \) is any vector, then \( D_{\overrightarrow{u}} f (x, y) = \nabla f (x, y) \cdot \frac{\overrightarrow{u}}{||\overrightarrow{u}||} \).

   True
3.8.5 Answers to Practice Problems

Do the following problems:

1. Find the domain, range and identify the level curves of \( f(x, y) = 9x^2 + y^2 \).
2. Find the domain, range and identify the level curves of \( f(x, y) = \sqrt{x^2 - y} \).
3. Find the domain, range and identify the level surfaces of \( f(x, y, z) = x^2 + y^2 - z \).
4. Find \( \lim_{(x,y) \to (\pi, \ln 2)} e^y \cos x \).
5. Find \( \lim_{(x,y) \to (1,1)} \frac{x - y}{x^2 - y^2} \).
6. By considering different paths of approach, show \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 - y} \) does not exist.
7. By considering different paths of approach, show \( \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{xy} \) does not exist.
8. Find all the first order partials for \( f(r, \theta) = r \cos \theta + r \sin \theta \).
9. Find all the first order partials for \( f(R_1, R_2, R_3) = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \).
10. Find all the second order partials for \( f(x, y) = y + \frac{x}{y} \).
11. Find all the second order partials for \( f(x, y) = e^x + y \sin x \).
12. Find all the second order partials for \( f(x, y) = x + xy - 5x^3 + \ln(x^2 + 1) \).
13. Find \( \frac{dw}{dt} \) at \( t = 0 \) if \( w = \sin(xy + \pi) \), \( x = e^t \) and \( y = \ln(t + 1) \).
14. Find \( \frac{\partial w}{\partial r} \) and \( \frac{\partial w}{\partial s} \) when \( r = \pi \) and \( s = 0 \) if \( w = \sin(2x - y) \), \( x = r + \sin s \) and \( y = rs \).
15. Find the directions in which \( f(x, y) = \cos x \cos y \) increases and decreases most rapidly at \( P_0 = \left( \frac{\pi}{4}, \frac{\pi}{4} \right) \) and find the rate of increase in each direction. Finally, find the derivative of \( f \) at \( P_0 \) in the direction of \( \vec{v} = (3, 4) \).
16. What is the largest value the directional derivative of \( f(x, y, z) = xyz \) can have at the point (1, 1, 1)?
17. Find an equation of the tangent plane and parametric equations of the normal line to the level surface \( x^2 - y - 5z = 0 \) at the point \( P_0 = (2, -1, 1) \).
18. Find an equation of the plane tangent to the surface \( z = \ln(x^2 + y^2) \) at the point \((0, 1, 0)\).

19. Find the linearization of \( f(x, y, z) = xy + 2yz - 3xz \) at the points \((1, 0, 0)\) and \((1, 1, 0)\).

20. Find the local extrema of \( f(x, y) = x^2 - xy + y^2 + 2x + 2y - 4 \). 
   \( f_x(x, y) = 2x - y + 2 \) and \( f_y(x, y) = -x + 2y + 2 \). These partials are always defined and are 0 when \( \begin{cases} 2x - y + 2 \\ -x + 2y + 2 \end{cases} \), Solution is: \([x = -2, y = -2]\).

21. Find the local extrema of \( f(x, y) = 2x^3 + 3xy + 2y^3 \).