4.4 Change of Variable in Integrals: The Jacobian

In this section, we generalize to multiple integrals the substitution technique used with definite integrals. For functions of two or more variables, there is a similar process we can use. It is a little bit more involved though. In addition, in higher dimensions, a change of variable can also be used to simplify the region of integration. We begin with an important definition.

4.4.1 The Jacobian

The Jacobian, named after the German mathematician **Carl Gustav Jacobi** (1804-1851), plays an important role in higher dimensional mathematics. In this section, we see how it is used when changing variables to simplify a region or an integrand. We begin with its definition.

Definition 352 (2-D case) *Suppose that x and y are two independent variables which can be expressed in term of two other independent variables u and v by the formula $x = g(u,v)$ and $y = h(u,v)$. **The Jacobian** of x and y with respect to u and v, denoted $\dfrac{\partial(x,y)}{\partial(u,v)}$ or $J(u,v)$, is*

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Definition 353 (3-D case) *Suppose that x, y and z are three independent variables which can be expressed in term of three other independent variables u, v and w by the formula $x = g(u,v,w)$, $y = h(u,v,w)$ and $z = l(u,v,w)$. **The Jacobian** of x, y and z with respect to u, v and w, denoted $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ or $J(u,v)$, is*

$$J(u,v) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Example 354 *Recall, when switching from Cartesian to polar coordinates, we have $x = r\cos\theta$ and $y = r\sin\theta$. The Jacobian of x and y with respect to r and*

θ is

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= r\cos^2\theta + r\sin^2\theta \\
\frac{\partial(x,y)}{\partial(r,\theta)} &= r
\end{aligned}$$

Example 355 *Recall, when switching from Cartesian to Spherical coordinates, we have $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$. The Jacobian of x, y, and z with respect to ρ, θ and ϕ is*

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} \\
&= \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}
\end{aligned}$$

The easiest way to compute this determinant is to expand it using the third row since one of its entries is 0. We obtain

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} &= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\
&= \cos\phi\left(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right) \\
&\quad -\rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right) \\
&= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin^3\phi \\
&= -\rho^2\sin\phi
\end{aligned}$$

4.4.2 Change of Variable

You will recall in one-dimensional calculus, when given an integral of the form $\int g'(u) f(g(u))\,du$, we performed the change of variable $x = g(u)$ which gave us $dx = g'(u)\,du$ and thus

$$\int g'(u) f(g(u))\,du = \int f(x)\,dx$$

We can write this slightly differently as follows. Since $x = g(u)$, $\dfrac{dx}{du} = g'(u)$ hence, we have

$$\int f(x)\,dx = \int f(g(u))\,\frac{dx}{du}du$$

The Jacobian is what generalizes $\dfrac{dx}{du}$ in the above formula. We begin with the change of variable theorem for double integrals. We then look at several examples to see how one can benefit from a change of variable. These benefits include using a change of variable to simplify an integrand, using a change of variable to simplify a region. As an application, we will look at double integrals in polar coordinates. Note that most of the explanations given below will be for regions in the xy-plane or for functions of two variables.

General Case

Let us first introduce some notation. Let R denote a region in the xy-plane and S a region in the uv-plane. A change of variable is usually described as a transformation T from the uv-plane to the xy-plane given by $T(u,v) = (x,y)$ where x and y are given by

$$\begin{aligned} x &= g(u,v) \\ y &= h(u,v) \end{aligned}$$

We usually assume that the first order partials of g and h are continuous. When it is the case we say that T is a C^1 **transformation**. Such a transformation will map a region S in the uv-plane into another region R into the xy-plane (see figure 4.4.2). In most cases, we are given R and are looking for a region S and a transformation T from S to R in which S is simpler than R. We will only consider simple cases here.

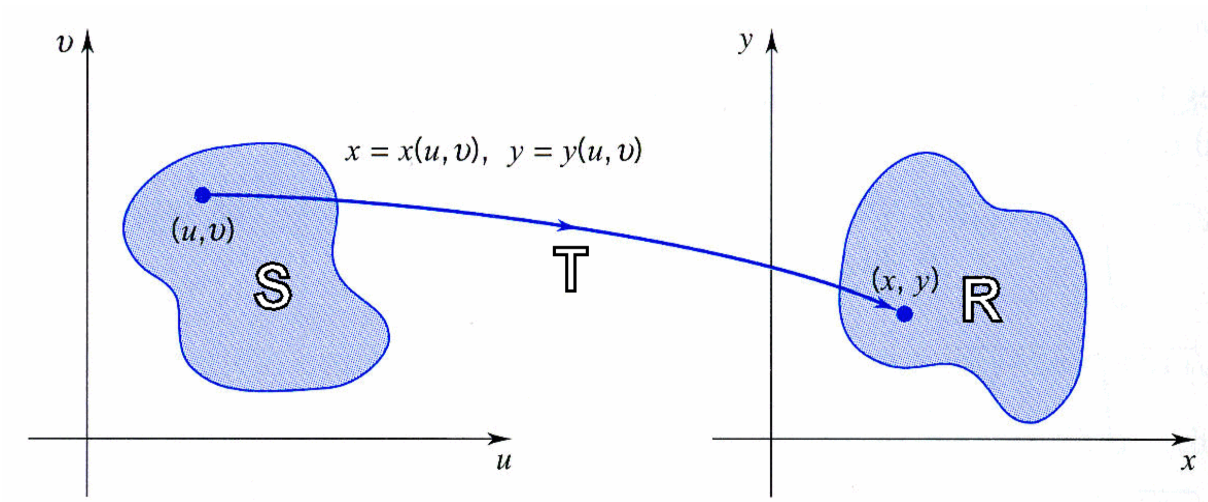
When we talk about the Jacobian of the transformation T, we mean the Jacobian of the change of variable

$$\begin{aligned} x &= g(u,v) \\ y &= h(u,v) \end{aligned}$$

We begin with the change of variable theorem for integrals, given without proof.

Theorem 356 (Change of Variable in Double Integrals) *Let R and S be regions of the xy-planes and uv-planes respectively. Let $T : S \to R$ be a C^1 transformation such that $T(u,v) = (x,y)$ where*

$$\begin{aligned} x &= g(u,v) \\ y &= h(u,v) \end{aligned}$$



such that each point in R is the image of a unique point in S. If f is continuous on R and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ on R then

$$\iint_R f(x,y)\,dxdy = \iint_S f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

Remark 357 *It is important to understand that R is given. We must find an appropriate change of variable in which the integral on the right is simpler. As stated, the theorem is deceiving. It makes it look like the integral on the left is easier than the integral on the right which appears to contain much more. But this is not the case in practice.*

Remark 358 *As in the one dimensional case, it is important to understand that when one performs a change of variable, not only the integrand changes, but also the region of integration. As mentioned already, there two possible reasons for performing a change of variable:*

1. *To obtain a simpler integrand.*
2. *To obtain an easier region over which to integrate. The ideal region is a rectangle with sides parallel to the coordinate axes. In this case, we can use Fubini's theorem.*

Remark 359 *Deciding what region works best for an integral takes practice. Finding the transformation T which maps one region into another is also very involved and could be the purpose of an entire course. Here, we will only consider simple examples. Problems will fall in two categories. The change of variable will be suggested or it will not.*

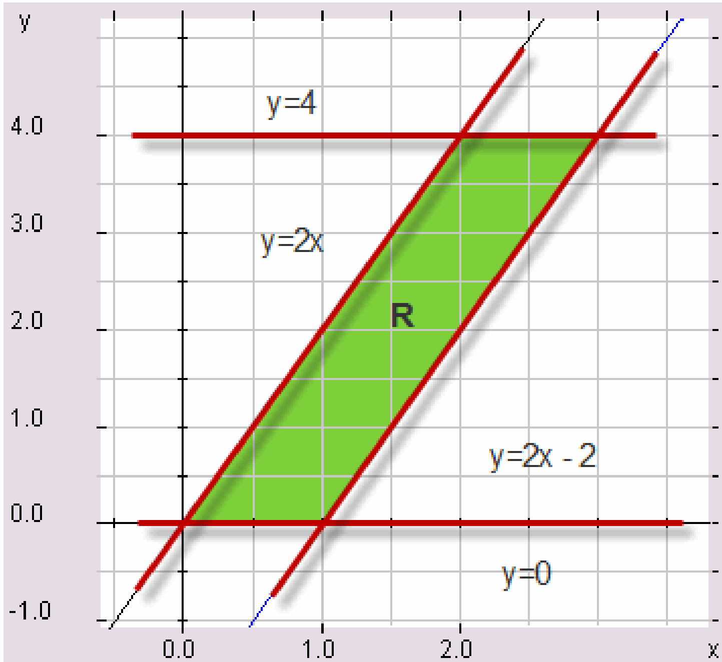


Figure 4.21: Region bounded by $y = 0$, $y = 4$, $y = 2x$ and $y = 2x - 1$

Example 360 *Evaluate* $\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2} dxdy$ *by applying the transformation* $u = \frac{2x-y}{2}$ *and* $v = \frac{y}{2}$.

The region corresponding to this integral is $R = \left\{(x,y) : 0 \leq y \leq 4, \frac{y}{2} \leq x \leq \frac{y}{2} + 1\right\}$. *This is a type II region. It is shown in figure 4.21. In the equations defining* u *and* v*, we need to solve for* x *and* y *since we need to compute the Jacobian of the transformation giving us* x*and* y *in terms of* u *and* v*. Simple algebra gives* $x = u + v$ *and* $y = 2v$*. Therefore*

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \\ &= 2 \end{aligned}$$

We also need to find what the new region, we call S *will be. It is enough to find its boundaries. We illustrate how to do this in the table below:*

xy equations for the boundary of R	*uv equations for the boundary of S*	*Simplified uv equations*
$x = \frac{y}{2}$	$u + v = \frac{2v}{2} = v$	$u = 0$
$x = \frac{y}{2} + 1$	$u + v = \frac{2v}{2} + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

Hence $S = \{(u,v) : 0 \leq u \leq 1, 0 \leq v \leq 2\}$. *Thisregion is shown in figure 4.22.*

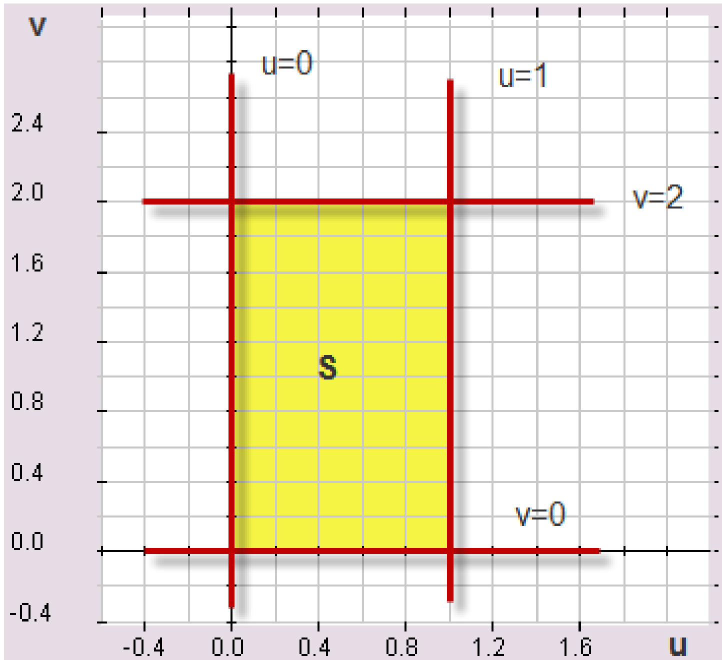


Figure 4.22: Region bouded by $u = 0$, $u = 1$, $v = 0$ and $v = 2$.

We are now ready to apply the change of variable formula.

$$\begin{aligned} \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x - y}{2} dxdy &= \int_0^2 \int_0^1 u \left| \frac{\partial (x, y)}{\partial (u, v)} \right| dudv \\ &= \int_0^2 \int_0^1 2ududv \\ &= \int_0^2 \left[u^2\right]\Big|_0^1 dv \\ &= \int_0^2 dv \\ &= 2 \end{aligned}$$

Example 361 *Evaluate* $\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\, dydx$ *using a transformation.*
First, let us find the region R *over which we are integrating. It is* $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$*. It is shown in figure 4.23. The integrand suggests the change of variable* $u = x+y$ *and* $v = y - 2x$*. First, we need to solve for* x *and* y*. Simple algebra suggests*

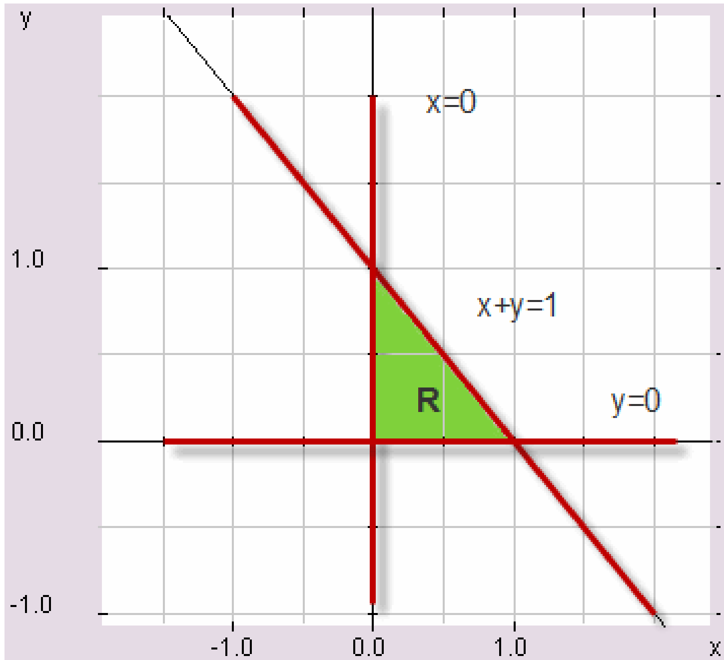


Figure 4.23: Region bounded by $x = 0$, $y = 0$, $x + y = 1$.

that $x = \dfrac{u - v}{3}$ *and* $y = \dfrac{2u + v}{3}$. *Next, we compute* $\dfrac{\partial (x, y)}{\partial (u, v)}$.

$$\begin{aligned} \frac{\partial (x, y)}{\partial (u, v)} &= \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} \\ &= \frac{1}{3} \end{aligned}$$

Finally, we need to find the corresponding uv region, we call it S. As in the previous example, we use a table.

xy equations for the boundary of R	*uv equations for the boundary of S*	*Simplified uv equations*
$x = 0$	$\dfrac{u - v}{3} = 0$	$v = u$
$y = 0$	$\dfrac{2u + v}{3} = 0$	$v = -2u$
$x + y = 1$	$\dfrac{u - v}{3} + \dfrac{2u + v}{3} = 1$	$u = 1$

So, we see that S is the region bounded by $u = 0$, $v = u$ *and* $v = -2u$. *It is shown in figure 4.24.We can write it as a type I region.* $S = \{(x, y) : 0 \leq u \leq 1, -2u \leq v \leq u\}$.

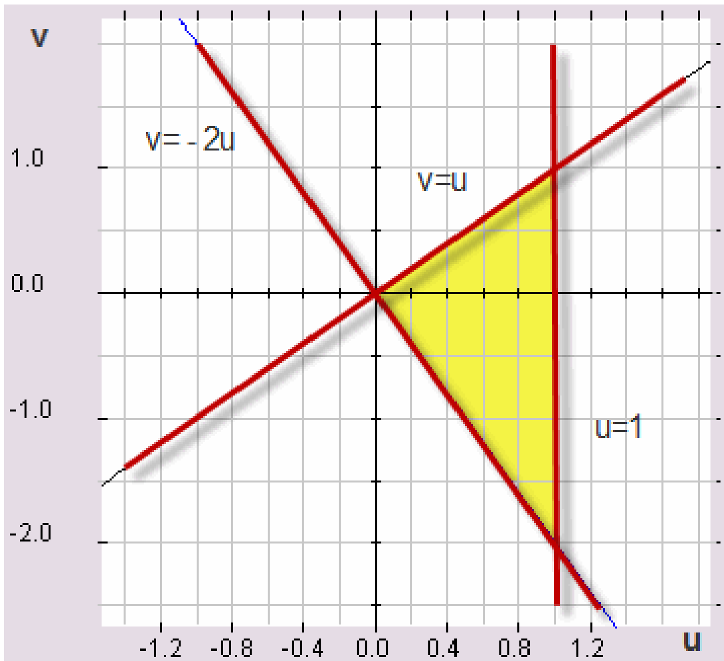


Figure 4.24: Region bounded by $u = 0$, $u = v$ and $v = -2u$.

We are now ready to do the change of variable.

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dydx &= \int_0^1 \int_{-2u}^{u} \sqrt{u}v^2 \left| \frac{\partial (x,y)}{\partial (u,v)} \right| dvdu \\
&= \int_0^1 \left[\frac{1}{3}\sqrt{u}\frac{v^3}{3} \right] \Big|_{-2u}^{u} du \\
&= \int_0^1 \left(\frac{u^{\frac{7}{2}}}{9} + \frac{8u^{\frac{7}{2}}}{9} \right) du \\
&= \int_0^1 u^{\frac{7}{2}}\,du \\
&= \frac{2}{9} u^{\frac{9}{2}} \Big|_0^1 \\
&= \frac{2}{9}
\end{aligned}$$

Example 362 *Evaluate* $\iint\limits_R (x+y)^2\,dxdy$ *where* R *is the region bounded by* $x+y=0$, $x+y=1$, $2x-y=0$, $y=3$ *by first performing a change of variable which maps a region* S *into* R *where* S *is a rectangle in the* uv*-plane.*

This region R is shown in figure 362. You will note that R is a parallelogram, that is opposite sides are parallel. This can be seen on figure 362 as well as in the equations of R. The variable part of these equations fall in two categories: $x+y$ *and* $2x-y$*. So, we only need to set* u *and* v *to these and solve for* x*. We let* $u=x+y$ *and* $v=2x-y$ *and solve for* x *and* y*.*

$$\begin{cases} u = x+y \\ v = 2x-y \end{cases}$$

Solving for y *in both equations gives* $y=u-x$ *and* $y=2x-v$ *thus*

$$\begin{aligned} u-x &= 2x-v \Longleftrightarrow u+v=3x \\ &\Longleftrightarrow x=\frac{u+v}{3} \end{aligned}$$

Using one of the expressions we got for y *gives*

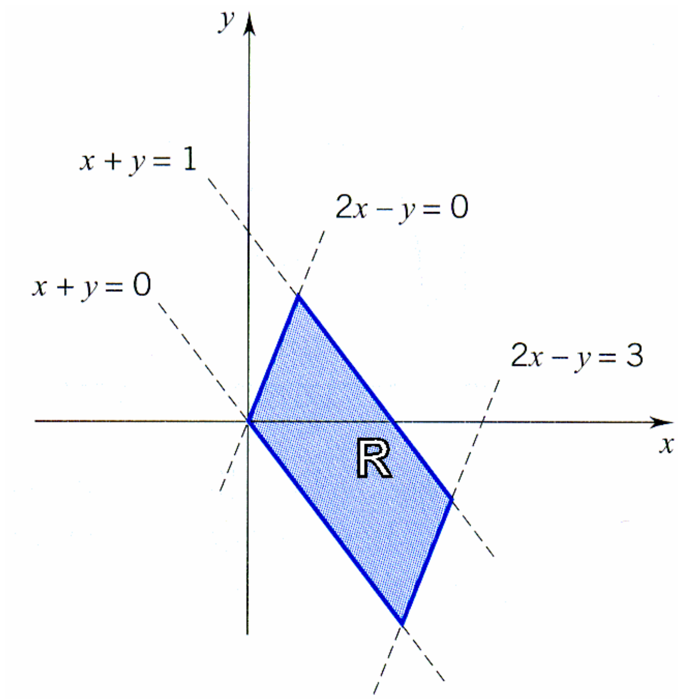
$$\begin{aligned} y &= u-x \\ &= u-\frac{u+v}{3} \\ &= \frac{2u-v}{3} \end{aligned}$$

We need to find the corresponding uv region we will call S *. As before, we do it with a table.*

xy equations for the boundary of R	*uv equations for the boundary of S*	*Simplified uv equations*
$x+y=0$	$u=0$	$u=0$
$x+y=1$	$u=1$	$u=1$
$2x-y=0$	$v=0$	$v=0$
$2x-y=3$	$v=3$	$v=3$

So $S=\{(x,y):0\leq u\leq 1 and 0\leq v\leq 3\}$*. It is a rectangle shown in figure 362. The transformation* T *(or change of variable) which takes points* (u,v) *in* S *to points* (x,y) *in* R *is* $T(u,v)=(x,y)$ *where*

$$\begin{cases} x=\dfrac{u}{3}+\dfrac{v}{3} \\ y=\dfrac{2u}{3}-\dfrac{v}{3} \end{cases}$$

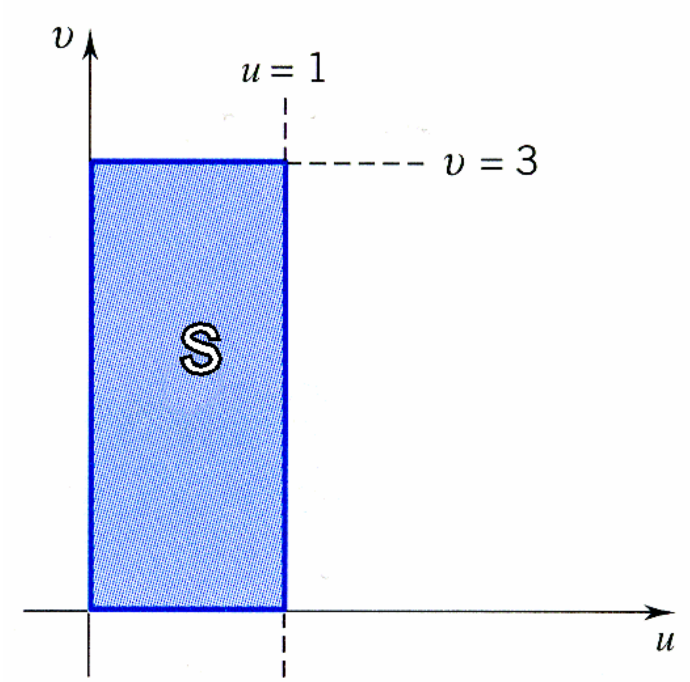


The Jacobian for this transformation is

$$\begin{aligned}
\frac{\partial (x,y)}{\partial (u,v)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{vmatrix} \\
&= \frac{-1}{9} - \frac{2}{9} \\
&= -\frac{1}{3}
\end{aligned}$$

Using the change of variable theorem ,we have

$$\begin{aligned}
\iint_R (x+y)^2 \, dxdy &= \iint_S f\left(g(u,v), h(u,v)\right) \left| \frac{\partial (x,y)}{\partial (u,v)} \right| dudv \\
&= \iint_S \frac{u^2}{3} dudv
\end{aligned}$$



Since S is a rectangular region, we can use Fubini's theorem to get

$$\begin{aligned}\iint_R (x+y)^2\,dxdy &= \iint_S \frac{u^2}{3}dudv \\ &= \frac{1}{3}\iint_S u^2 dudv \\ &= \frac{1}{3}\int_0^3 dv \int_0^1 u^2 du \\ &= \frac{1}{3}(3)\left(\frac{1}{3}\right) \\ &= \frac{1}{3}\end{aligned}$$

Example 363 *Evaluate* $\iint_R (x+y)^2 \sin\left({}^2x-y\right) dA$ *if R is the region bounded by the square with vertices $(0,1)$, $(1,2)$, $(2,1)$ and $(1,0)$*
The region R is shown in figure 363. Note that the sides of R lie on the lines $x-y=1$, $x-y=-1$, $x+y=1$ and $x+y=3$. As above, we let $u=x+y$ and $v=x-y$. It is easy to determine the bounds of u and v for the S region. From the boundaries of R, we see that $S=\{(u,v): 1\le u\le 3$ and $-1\le v\le 1\}$. We

need to express x and y in terms of u and v so we can compute the Jacobian. Since

$$\begin{cases} u = x + y \\ v = x - y \end{cases}$$

solving for y gives us $y = u - x$ *and* $y = x - v$ *thus*

$$\begin{aligned} u - x &= x - v \Longleftrightarrow u + v = 2x \\ &\Longleftrightarrow x = \frac{u}{2} + \frac{v}{2} \end{aligned}$$

replacing in one of the expressions for y gives

$$\begin{aligned} y &= u - x \\ &= u - \frac{u}{2} - \frac{v}{2} \\ &= \frac{u}{2} - \frac{v}{2} \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} \\ &= \frac{-1}{4} - \frac{1}{4} \\ &= \frac{-1}{2} \end{aligned}$$

Using the change of variables theorem, we see that

$$\begin{aligned} \iint_R (x+y)^2 \sin^2(x-y)\, dA &= \iint_S f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \\ &= \iint_S u^2 \sin^2 v \left(\frac{1}{2}\right) dudv \end{aligned}$$

Since S is a rectangular region, by Fubini's theorem, we get

$$\begin{aligned} \iint_R (x+y)^2 \sin^2(x-y)\, dA &= \iint_S u^2 \sin^2 v \left(\frac{1}{2}\right) dudv \\ &= \frac{1}{2} \int_{-1}^{1} \int_{1}^{3} u^2 \sin^2 v \left(\frac{1}{2}\right) dudv \\ &= \frac{1}{2} \int_{-1}^{1} \sin^2 v dv \int_{1}^{3} u^2 du \end{aligned}$$

We can do each integral separately.

$$\begin{aligned}\int_1^3 u^2 du &= \left.\frac{u^3}{3}\right|_1^3 \\ &= 9 - \frac{1}{3} \\ &= \frac{26}{3}\end{aligned}$$

and, using the fact that $sin^2 x = \dfrac{1-\cos 2x}{2}$

$$\begin{aligned}\int_{-1}^{1} \sin^2 v dv &= \frac{1}{2}\int_{-1}^{1} (1-\cos 2v)\, dv \\ &= \frac{1}{2}\left.\left(v - \frac{1}{2}\sin 2v\right)\right|_{-1}^{1} \\ &= \frac{1}{2}\left(\left(1-\frac{1}{2}\sin 2\right) - \left(-1-\frac{1}{2}\sin(-2)\right)\right) \\ &= 1 - \frac{1}{2}\sin 2\end{aligned}$$

Therefore

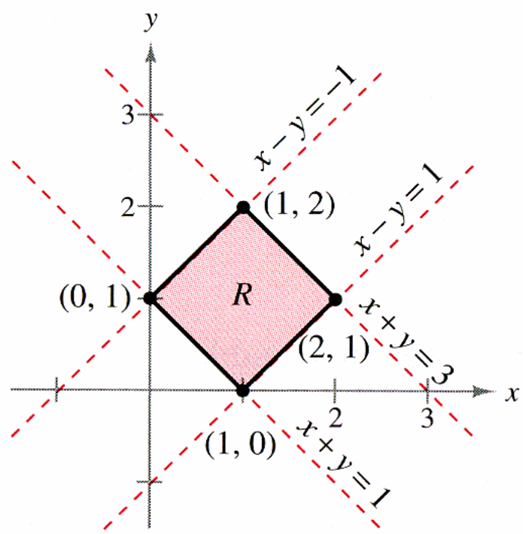
$$\begin{aligned}\iint_R (x+y)^2 \sin^2(x-y)\, dA &= \frac{1}{2}\left(\frac{26}{3}\right)\left(1-\frac{1}{2}\sin 2\right) \\ &\approx 2.363\,2\end{aligned}$$

Application of Change of Variable to Polar Coordinates

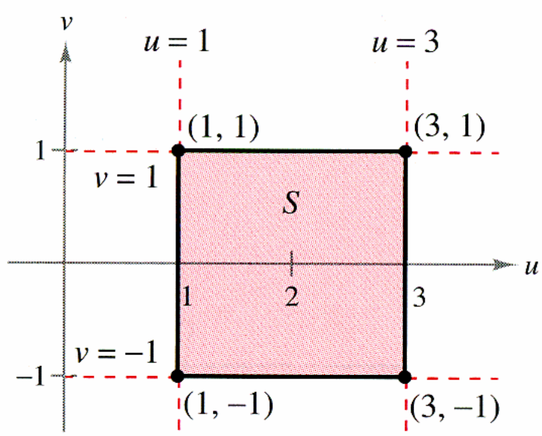
You will recall that the change of variable from Cartesian to polar coordinates is $x = r\cos\theta$ and $y = r\sin\theta$. Also, from example 354, the Jacobian of this transformation is $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r$. Therefore, from the change of variable theorem, we have

$$\begin{aligned}\iint_R f(x,y)\, dxdy &= \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta \\ &= \iint_S f(r\cos\theta, r\sin\theta)\, rdrd\theta\end{aligned}$$

where S is the region in the $r\theta$-plane corresponding to R in the xy-plane.



Region R in the xy-plane



Region S in the uv-plane

Example 364 *Evaluate* $\iint\limits_R e^{x^2+y^2}dxdy$ *where* R *is the upper half portion of the unit circle.*
The region R *can be written as* $R = \left\{(x,y) : -1 \le x \le 1 \text{ and } 0 \le y \le \sqrt{1-x^2}\right\}$*. The corresponding region* S *is* $S = \{(r,\theta) : 0 \le r \le 1 \text{ and } 0 \le \theta \le \pi\}$*. Therefore*

$$\begin{aligned}\iint\limits_R e^{x^2+y^2}dxdy &= \iint\limits_S re^{r^2}drd\theta \\ & \int_0^\pi \int_0^1 re^{r^2}drd\theta\end{aligned}$$

We can evaluate the inner integral using the substitution $u = r^2$ *hence* $du = 2rdr$ *thus*

$$\begin{aligned}\int_0^1 re^{r^2}dr &= \frac{1}{2}\int_0^1 e^u du \\ &= \frac{1}{2}(e-1)\end{aligned}$$

You will note that in the process, we updated the limits of integration. It does not sow because they end up being the same. But make sure not to forget to do it. The original limits were 0 *and* 1*, they were limits for* r*. When we substitute and use* u *instead, we must find the limits on* u*. Since* $u = r^2$ *when* $r = 0$*,* $u = 0^2 = 0$ *and when* $r = 1$*,* $u = 1^2 = 1$*. We can now finish the integral.*

$$\begin{aligned}\iint\limits_R e^{x^2+y^2}dxdy &= \int_0^\pi \frac{1}{2}(e-1)\,d\theta \\ &= \frac{\pi}{2}(e-1)\end{aligned}$$

Example 365 *Evaluate* $\iint\limits_R \left(x^2+y\right)dA$ *where* R *is the annular region between the two circles* $x^2+y^2=1$ *and* $x^2+y^2=5$*. The corresponding region* S *is*

$$S = \left\{(r,\theta) \mid 1 \le r \le \sqrt{5} \text{ and } 0 \le \theta \le 2\pi\right\}$$

Therefore

$$\begin{aligned}\iint\limits_R \left(x^2+y\right)dA &= \int_0^{2\pi}\int_1^{\sqrt{5}} \left(r^2\cos^2\theta + r\sin\theta\right) r dr d\theta \\ &= \int_0^{2\pi}\int_1^{\sqrt{5}} \left(r^3\cos^2\theta + r^2\sin\theta\right) dr d\theta \\ &= \int_0^{2\pi} \left(\frac{r^4}{4}\cos^2\theta + \frac{r^3}{3}\sin\theta\right)\Bigg|_1^{\sqrt{5}} d\theta \\ &= \int_0^{2\pi}\left(6\cos^2\theta + \frac{5\sqrt{5}-1}{3}\sin\theta\right) d\theta\end{aligned}$$

Using the fact that $\cos^2\theta = \dfrac{1+\cos 2x}{2}$*, we have*

$$\begin{aligned}\iint\limits_R \left(x^2+y\right)dA &= \int_0^{2\pi}\left(3+3\cos 2\theta + \frac{5\sqrt{5}-1}{3}\sin\theta\right) d\theta \\ &= 6\pi\end{aligned}$$

4.4.3 Assignment

1. This section contains more material than what you need to know for the final. I want you to know the following:
   (a) The Jacobian. Given a transformation, be able to find the Jacobian corresponding to it.
   (b) Being able to evaluate integrals using polar coordinates.
2. Solve the system $\left\{\begin{array}{c} u = x - y \\ v = 2x + y \end{array}\right.$ for x and y then find $\dfrac{\partial(x,y)}{\partial(u,v)}$.
3. Solve the system $\left\{\begin{array}{c} u = x + 2y \\ v = x - y \end{array}\right.$ for x and y then find $\dfrac{\partial(x,y)}{\partial(u,v)}$.
4. Rewrite the given integrals from Cartesian to polar coordinates, then evaluate the integral.
   (a) $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} dydx$
   (b) $\int_0^1\int_0^{\sqrt{1-x^2}} \left(x^2+y^2\right) dydx$
   (c) $\int_{-a}^{a}\int_{-\sqrt{a-x^2}}^{\sqrt{a-x^2}} dydx$
   (d) $\int_0^6\int_0^y xdxdy$