2.3 Cramer’s Rule

2.3.1 Theory

Cramer’s rule is mostly needed for a variety of theoretical calculations. It is inefficient for hand calculations, except for calculations involving $2 \times 2$ or $3 \times 3$ matrices. In this case, it provides a formula for the solution of systems of $2$ equations in $2$ unknowns or systems of $3$ equations in $3$ unknowns.

Let us first introduce some notation. Given an $n \times n$ matrix $A$ and an $n \times 1$ matrix $b$, let $A_i(b)$ denote the $n \times n$ matrix obtained by replacing the $i^{th}$ column of $A$ by $b$.

Example 172 If $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 5 & 3 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 10 \\ 12 \\ 20 \end{bmatrix}$ then

$A_1(b) = \begin{bmatrix} 10 & 3 & 2 \\ 12 & 1 & 1 \\ 20 & 3 & 1 \end{bmatrix}$

$A_2(b) = \begin{bmatrix} 1 & 10 & 2 \\ 2 & 12 & 1 \\ 5 & 20 & 1 \end{bmatrix}$

and

$A_3(b) = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 1 & 12 \\ 5 & 3 & 20 \end{bmatrix}$

Theorem 173 (Cramer’s Rule) Let $A$ be an $n \times n$ invertible matrix. For any $n \times 1$ matrix $b$, the unique solution of $Ax = b$ where $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is given by

$$x_i = \frac{|A_i(b)|}{|A|} \text{ for } i = 1, 2, ..., n$$ (2.4)

Proof. Let $a_1, a_2, ..., a_n$ denote the columns of $A$ and $e_1, e_2, ..., e_n$ denote the columns of $I_n$ in other words $A = [a_1, a_2, ..., a_n]$ and $I_n = [e_1, e_2, ..., e_n]$. Then,

$$AI_i(x) = A [e_1, e_2, ..., e_{i-1}, x, e_{i+1}, ..., e_n]$$

$$= [Ae_1, Ae_2, ..., Ae_{i-1}, Ax, Ae_{i+1}, ..., Ae_n]$$

$$= [a_1, a_2, ..., a_{i-1}, b, a_{i+1}, ..., a_n]$$

$$= A_i(b)$$
Taking the determinant, and using the properties of determinants, we get

\[
\begin{align*}
|AI_i(x)| &= |A_i(b)| \\
|A| |I_i(x)| &= |A_i(b)| \\
|A|x_i &= |A_i(b)|
\end{align*}
\]

and thus

\[
x_i = \frac{|A_i(b)|}{|A|}
\]

Example 174 Solve \( \begin{cases} 3x_1 - 2x_2 = 6 \\
-5x_1 + 4x_2 = 8 \end{cases} \)

View the system as the matrix equation \( Ax = b \) where \( A = \begin{bmatrix} 3 & -2 \\
-5 & 4 \end{bmatrix}, x = \begin{bmatrix} x_1 \\
x_2 \end{bmatrix} \) and \( b = \begin{bmatrix} 6 \\
8 \end{bmatrix}. \) Then, \( A_1(b) = \begin{bmatrix} 6 & -2 \\
8 & 4 \end{bmatrix} \) and \( A_2(b) = \begin{bmatrix} 3 & 6 \\
-5 & 8 \end{bmatrix}. \)

Therefore,

\[
x_1 = \frac{\begin{vmatrix} 6 & -2 \\
8 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\
-5 & 4 \end{vmatrix}} = \frac{40}{2} = 20
\]

and

\[
x_2 = \frac{\begin{vmatrix} 3 & 6 \\
-5 & 8 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\
-5 & 4 \end{vmatrix}} = \frac{54}{2} = 27
\]

2.3.2 Concepts Review

- Know what Cramer’s rule is.
- Be able to use it to solve systems of linear equations.

2.3.3 Problems

Do # 16, 17, 18, 23 on page 95.