1.4 More Matrix Operations and Properties

In this section, we look at the properties of the various operations on matrices. As we do so, we try to draw a parallel between matrices and real numbers and the properties of the operations we use on them. We will see that there are many similarities, but also important differences. It is important to understand these differences. Some techniques we use with real numbers will not work with matrices.

1.4.1 Properties of the Standard Matrix Operations

Having defined matrices, and some of the operations which can be performed on them, it is important to know the properties of each operation so we know how to manipulate matrices with these operations.

We will prove some of the properties given in this section to illustrate the proofs techniques used. The remaining proofs will be left as exercises. As we do these proofs, it will be important to remember the definitions and various theorems studied. Some proofs will involve using the definitions and working on the entries of matrices. Other proofs will involve using results already derived.

Properties of addition and subtraction

With these two operations, matrices behave very much like real numbers. They have the same properties.

**Proposition 69** We assume that the sizes of the matrices involved are such that the operations listed are possible. The set of $m \times n$ matrices with real coefficients together with addition is an **Abelian (commutative) group**. That is, addition satisfies the following properties:

1. **Addition is commutative** that is $A + B = B + A$ for any two matrices $A$ and $B$ in the set.

2. **Addition is associative** that is $A + (B + C) = (A + B) + C$ for any matrices $A, B, C$ in the set.

3. There exists an **additive identity matrix**, the $m \times n$ matrix whose entries are all 0’s. If we denote this matrix by 0, then it has the following property: $A + 0 = 0 + A = A$.

4. Each matrix has an additive inverse. The additive inverse of $A$ is $-A$. It satisfies: $A + (-A) = -A + A = 0$, where 0 is the zero matrix here.

**Proof.** Since matrix addition/subtraction amounts to adding/subtracting corresponding entries, these properties will follow from the same properties of real numbers. We prove part 1 and leave the other parts as exercises.
1. Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $(A + B)_{ij}$ denote the $ij$ entry of $A + B$. We need to prove that $(A + B)_{ij} = (B + A)_{ij}$. By definition,

$$
(A + B)_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} \quad \text{(addition of real numbers is commutative)}
$$

$$
= (B + A)_{ij}
$$

2. See exercises.

3. See exercises.

4. See exercises.

\begin{remark}
Properties 2-4 are the properties of a group.
\end{remark}

\begin{remark}
You should notice the similarity between the properties of matrices and real numbers.
\end{remark}

\begin{properties of scalar multiplication}
\begin{proposition}
We assume that the sizes of the matrices involved are such that the operations listed are possible. The set of $m \times n$ matrices with real coefficients together with scalar multiplication satisfies the following properties:

1. $1A = A1 = A$ for every matrix $A$ in the set.

2. $(c_1c_2)A = c_1(c_2A)$ for every scalar $c_1, c_2$ and every matrix $A$ in the set.

3. $c(A + B) = cA + cB$ for every scalar $c$ and every matrix $A$ and $B$ in the set.

4. $(c_1 + c_2)A = c_1A + c_2A$ for every scalar $c_1, c_2$ and every matrix $A$ in the set.
\end{proposition}
\end{properties of scalar multiplication}

\begin{remark}
Propositions 69 and 72 imply that the set of $m \times n$ matrices with real coefficients together with addition and scalar multiplication is a vector space. We will study vector spaces in greater details in the chapters to come.
\end{remark}

\begin{properties of matrix multiplication}
Several important properties real numbers have with multiplication are not shared by matrices.

We begin with the identity element for matrix multiplication, called the identity matrix. It is a diagonal matrix with 1's on its diagonal. It plays a role similar to 1 for multiplication of real numbers. Of course, there is an $n \times n$ identity matrix for each value of $n$.


Proposition 74 Suppose that $A$ is $m \times n$ and recall that $I_n$ denotes the $n \times n$ identity matrix (diagonal matrix with 1's on the diagonal). Then,

$$AI_n = A$$

and

$$I_mA = A$$

You will note that a different identity matrix was used (why?).

**Proof.** Two matrices are equal when their corresponding entries are equal. We begin by introducing some notation. Let $A = [a_{ij}]$, $I_n = [b_{ij}]$. Let $[c_{ij}] = (AI_n)_{ij}$, the $ij$ entry of $AI_n$. We need to show that $c_{ij} = a_{ij}$. We know by definition that $b_{ij} = 0$ if $i \neq j$ and $b_{ii} = 1$. From the definition of the product of two matrices, we have

$$c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$$

$$= a_{ij} \quad \text{(why?)}$$

The second equality is proved the same way. ■

Proposition 75 Let $A$, $B$ and $C$ be matrices with sizes such that the operations below are possible. Then, the following properties hold:

1. **Matrix multiplication is associative**, that is $A(BC) = (AB)C$.
2. With 0 denoting the zero matrix, $0A = A0 = 0$.
3. **Left distributive law**: $A(B + C) = AB + AC$.
4. **Right distributive law**: $(B + C)A = BA + CA$.

There are important properties which hold for real numbers but not for matrices. We list the most important ones.

1. **Matrix multiplication is not commutative**, though it is possible to find some matrices for which their product will be. This is easy to see. For example, if $A$ is $m \times n$ and $B$ is $n \times m$, then $AB$ is $m \times m$ and $BA$ is $n \times n$. $AB$ and $BA$ do not even have the same size.

2. You may then ask, how about if $A$ and $B$ are such that $AB$ and $BA$ have the same size, for example both $A$ and $B$ could be $n \times n$. In most cases, $AB$ and $BA$ will still not be equal. Here is an example.

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 5 & 6 \end{bmatrix}$$

and

$$\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}$$

So, the two are not equal.
3. **The cancellation law does not hold.** Recall for real numbers if \( ab = ac \) and \( c \neq 0 \) then \( a = b \). This is used a lot when solving equations. This does not hold for matrices. Consider

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}
\]

\[
B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}
\]

and

\[
C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}
\]

The reader can verify that

\[
AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}
\]

and

\[
AC = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}
\]

So that \( AB = AC \) yet \( A \neq B \).

4. Another important property of real numbers is the principle of zero products which states that the product of two real numbers is zero if and only if at least one of the factors is zero. **It is possible for two non zero matrices to have a product equal to zero.** Here is an example:

\[
\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

We finish with an important theorem concerning the reduced row-echelon form of a matrix.

**Theorem 76** The reduced row-echelon form of an \( n \times n \) matrix either has at least a row of zeroes or is the identity matrix.

**Proof.** This is an either or result. A common technique is to assume one of the conclusion is not true and show the other has to happen. If we assume the reduced row-echelon form has no rows consisting entirely of zeroes, then every row has a leading one. Since the matrix is \( n \times n \), there will be \( n \) leading ones, staggered further to the right as we move down the matrix. Since no two rows can be the same, these leading ones have to be along the diagonal. ■
1.4.2 Multiplicative Inverse of a Matrix

Keeping with the parallel between real numbers and matrices, we know that every real number not equal to 0 has a multiplicative inverse. For matrices, it is not as simple. First, only square matrices have an inverse. Second, not every square matrix has an inverse. We begin with the definition of the inverse of a matrix.

**Definition 77** Let \( A \) be an \( n \times n \) matrix. If there exists a matrix \( B \), also \( n \times n \) such that

\[
AB = BA = I_n
\]

then \( B \) is called the multiplicative inverse of \( A \). The multiplicative inverse of a matrix \( A \) is usually denoted \( A^{-1} \).

**Remark 78** Note that the above definition also says that \( A \) is the inverse of \( B \).

Its is important to note that we only talk about inverses for square matrices. It is also important to understand that not every square matrix has an inverse.

**Proposition 79** If a matrix \( A \) has an inverse, then it is unique.

**Proof.** See exercises at the end of the section.

**Example 80** The inverse of

\[
\begin{bmatrix}
1 & 2 & 3 \\
3 & 2 & 1 \\
1 & 3 & 6
\end{bmatrix}
\]

is

\[
\begin{bmatrix}
-\frac{9}{7} & \frac{3}{4} & 1 \\
\frac{12}{7} & -\frac{3}{4} & -2 \\
-\frac{7}{4} & \frac{1}{4} & 1
\end{bmatrix}
\]

To check this, we compute

\[
\begin{bmatrix}
1 & 2 & 3 \\
3 & 2 & 1 \\
1 & 3 & 6
\end{bmatrix}
\begin{bmatrix}
-\frac{9}{7} & \frac{3}{4} & 1 \\
\frac{12}{7} & -\frac{3}{4} & -2 \\
-\frac{7}{4} & \frac{1}{4} & 1
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

and

\[
\begin{bmatrix}
-\frac{9}{7} & \frac{3}{4} & 1 \\
\frac{12}{7} & -\frac{3}{4} & -2 \\
-\frac{7}{4} & \frac{1}{4} & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 6
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

**Example 81** The matrix \( A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \) does not have an inverse. We see this by showing that no matrix can be the inverse of \( A \). If it did, it would also be a \( 2 \times 2 \) matrix. Let \( B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). Then, we should have \( AB = BA \) as

\[
BA = \begin{bmatrix}
1 & 0 \\ 0 & 1
\end{bmatrix}
\]

However, no matter what the entries of \( B \) are, we have

\[
BA = \begin{bmatrix}
a & b \\ c & d
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\ 3 & 0
\end{bmatrix} = \begin{bmatrix} a+3b & 0 \\ c+3d & 0 \end{bmatrix}
\]

Since the second column consists of zeroes, \( BA \) can never be the identity matrix.
Finding the inverse of a matrix is a long and tedious process. In the next sections, we will develop a technique to do so. For $2 \times 2$ matrices, there is an easy answer. We give it as a theorem.

**Theorem 82** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $A$ is invertible if $ad - bc \neq 0$. In this case,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Proof.** We leave it to the reader to verify that $AA^{-1} = I$.

**Example 83** The inverse of $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ is $\frac{1}{2} \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$. We can check it by computing the product of the two.

$$\frac{1}{2} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\frac{1}{2} \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We finish this subsection with a theorem regarding the inverse of a product of matrices.

**Theorem 84** If $A$ and $B$ are two invertible matrices having the same size, then $$(AB)^{-1} = B^{-1}A^{-1}$$

**Proof.** We need to show that $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} \quad \text{(associativity)}$$

$$= AA^{-1}$$

$$= AA^{-1}$$

$$= I$$

Similarly

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$$

$$= B^{-1}IB$$

$$= B^{-1}B$$

$$= I$$

In fact, this result can be extended to a product of any number of matrices. The inverse of a product of matrices is the product of their inverses, in the reverse order.
1.4.3 Power of a Matrix

Like with real numbers, we can define the power of a square matrix.

**Definition 85** If $A$ is a square matrix, and $n$ is a positive integer, then we define the following:

1. $A^0 = I$
2. $A^n = AA \cdots A$ ($n$ factors).
3. If $A$ is invertible, then $A^{-n} = (A^{-1})^n$

**Example 86** Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. Then

\[
A^2 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}
\]

\[
A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix}
\]

**Example 87** Let $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then

\[
B^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}
\]

\[
B^3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}
\]

Can you guess, then prove what $B^n$ is?

The laws of exponents which hold for real numbers also hold for matrices. We give them as a theorem.
Theorem 88  If $A$ is a square matrix and $r$ and $s$ are two positive integers then
\[ A^r A^s = A^{r+s} \]

**Proof.** See problems. 

Theorem 89  If $A$ is an invertible square matrix then:

1. $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.
2. $A^n$ is invertible and $(A^n)^{-1} = A^{-n}$ for $n = 0, 1, 2, \ldots$
3. For any nonzero scalar $k$, $(kA)^{-1} = \frac{1}{k} A^{-1}$.

**Proof.** We prove each result separately.

1. Since $A$ has an inverse which is $A^{-1}$, we have: $AA^{-1} = A^{-1}A = I$. This also says that $A$ is the inverse of $A^{-1}$ in other words $(A^{-1})^{-1} = A$. Since the inverse is unique, the result follows.

2. See problems.

3. We prove it by verification.

\[
(kA) \left( \frac{1}{k} A^{-1} \right) = \frac{1}{k} AA^{-1} \\
= AA^{-1} \\
= I
\]

and

\[
\left( \frac{1}{k} A^{-1} \right) (kA) = \frac{1}{k} kA^{-1} A \\
= A^{-1} A \\
= I
\]

We can use these definitions and properties to extend to matrices concepts which apply to real numbers. We give two as examples.

**Example 90 (Matrix Polynomial)**  Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. Let $A$ be an $n \times n$ matrix. We define

\[ p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n \]
Example 91 (Exponential of a Matrix) Recall from Calculus II that the Taylor series for $e^x$ is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

As we did above, we define

$$e^A = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots$$

The exponential of a matrix has applications in solving differential equations.

1.4.4 Properties of the Transpose

Theorem 92 Let $A$ and $B$ be two matrices with sizes such that the operations below are possible. Let $k$ be any scalar. Then, the following is true:

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$
5. If $A$ is also invertible, then $(A^T)^{-1} = (A^{-1})^T$

Proof. We prove some of these results, the remaining ones are left as exercises.

1. Recall that if $A = [a_{ij}]$ then $(A^T)_{ij} = a_{ji}$ in other words, we switch the rows and columns which amounts to switching the indices. We want to show that $(A^T)^T_{ij} = a_{ij}$. If $A = [a_{ij}]$ then $(A^T)_{ij} = a_{ji}$ (switch indices). Therefore, $(A^T)^T_{ij} = a_{ij}$ (switch indices again).

2. See problems

3. See problems

4. This one is a little bit more tricky, we show the proof. Let $A = [a_{ij}]$ be $m \times r$ and $B = [b_{ij}]$ be $r \times n$. To prove that $(AB)^T = B^T A^T$ we need to prove the matrices have the same size and the same corresponding entries.

   - **Same size.** $AB$ will be an $m \times n$ matrix, so $(AB)^T$ will be an $n \times m$ matrix. $B^T$ will be an $n \times r$ matrix and $A^T$ will be an $r \times m$ matrix so that $B^T A^T$ is an $n \times m$ matrix. Thus $(AB)^T$ and $B^T A^T$ have the same size.
• **Same corresponding entries.** Let \((AB)^T\) denote the \(ij\) entry of \((AB)^T\) and \((B^T A^T)_{ij}\) denote the \(ij\) entry of \(B^T A^T\). We need to prove that \((AB)^T\) = \((B^T A^T)_{ij}\). We have
\[
(AB)^T_{ij} = (AB)_{ji} = a_{j1}b_{1i} + a_{j2}b_{2i} + \ldots + a_{jr}b_{ri}
\]
If we call \((A^T)_{ij} = a'_{ij}\) and \((B^T)_{ij} = b'_{ij}\), we also have
\[
(B^T A^T)_{ij} = b'_{i1}a'_{1j} + b'_{i2}a'_{2j} + \ldots + b'_{ir}a'_{rij} = b_{1i}a_{j1} + b_{2i}a_{j2} + \ldots + b_{ri}a_{jr} = a_{j1}b_{1i} + a_{j2}b_{2i} + \ldots + a_{jr}b_{ri}
\]
So, we see the two are equal.

5. **We prove it by verification.** We need to show that \((A^{-1})^T A^T = A^T (A^{-1})^T = I\).
\[
(A^{-1})^T A^T = (AA^{-1})^T \text{ by } \#4
\]
\[
= I^T
\]
\[
= I
\]
and
\[
A^T (A^{-1})^T = (A^{-1}A)^T
\]
\[
= I^T
\]
\[
= I
\]

**Remark 93** The result of \#4 can be extended to a product of more than two matrices. The transpose of a product of matrices is the product of their transpose in reverse order.

### 1.4.5 Matrix Equations

As mathematical objects, matrices can appear in equations the same way numbers do. Equations involving matrices are solved in a similar way. In this section, we only look at equations of the form

\[ Ax = b \]

where \(A\) is \(n \times n\), \(x\) is \(n \times 1\) and \(b\) is \(n \times 1\). Solving the equation means finding \(x\) such that the equation is satisfied.

When solving matrix equations, the following operations are permitted:
1. Add the same matrix on each side of the equation.

2. Multiply each side of the equation by the same non-zero scalar.

3. Multiply each side of the equation by the same non-zero matrix.

Let us assume for now that $A$ has an inverse. Then, to solve $Ax = b$, we proceed as follows:

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b \quad \text{(multiply each side by the same matrix)}$$

$$I_nx = A^{-1}b \quad \text{(use the fact that } A^{-1}A = I_n)$$

$$x = A^{-1}b \quad \text{(property of the identity matrix)}$$

Of course, the above assumes that $A$ has an inverse.

**Application: Linear System of Equation Written as a Matrix Equation**

One technique used to solve a linear system of $n$ equations in $n$ variables is to write it as a matrix equation, then solve the matrix equation. It is easy to see that the above system is equivalent to the matrix equation

$$Ax = b$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$A$ is called the coefficient matrix, $b$ is called the constant matrix. Therefore, the solution, when it exists, is

$$x = A^{-1}b$$

It should be noted that this is not the preferred way to solve a system. Finding the inverse of a matrix is difficult and time consuming. There are better methods.
Remark 94 With the notation of this section, the augmented matrix for a system can be denoted

\[
\begin{bmatrix}
A & b \\
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\
\vdots & \vdots & \ddots & \vdots & | & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & | & b_n \\
\end{bmatrix}
\]

1.4.6 Problems on Sections

1. Finish proving proposition 74.

2. Prove proposition 79 that is prove that if a matrix has an inverse, then the inverse is unique. Hint: assume there are two inverses and show they are equal.

3. Finish proving the properties in propositions 69 and 72.

4. Finish proving proposition 75.

5. Think of problems in your areas of interest in which solving the problem would involve solving a linear system of equations.

6. Find \( B^n \) for any positive integer \( n \) for \( B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \). You must prove your claim.


8. Finish proving theorem 89.
