3.3 Integration and Differentiation of Fourier Series

3.3.1 Goal

When doing manipulations with infinite sums, we must remember that many properties which hold for finite sums do not always hold for infinite sums. In this section, we will see that the derivative of a sum (infinite) is not always the sums of the derivatives. In other words, we cannot always differentiate a Fourier series term by term. In this section, we discuss continuity, differentiation and integration of Fourier series.

3.3.2 Linearity of Fourier Series

Theorem 144 (Linearity of Fourier Series) Suppose that \( f_1 \) and \( f_2 \) are piecewise smooth on \([-L,L]\) and \( c_1 \) and \( c_2 \) are two constants. Then, the Fourier series of \( c_1f_1(x) + c_2f_2(x) \) is \( c_1 \) times the Fourier series of \( f_1(x) \) plus \( c_2 \) times the Fourier series of \( f_2(x) \).

The theorem means that if for example we know the Fourier series of \( x \) and \( x^2 \) then the Fourier series of \( x + x^2 \) is the sum of the Fourier series of \( x \) and \( x^2 \). We don’t have to go through the lengthy process of computing Euler’s coefficients.

3.3.3 Continuity of Fourier Series

From the convergence theorem on Fourier series (theorem 143), we know that where \( f \) is continuous, its Fourier series converges to \( f \) and is therefore continuous. The only points at which we need to worry about are the points where we have jump discontinuities. This can happen at points where the function itself has jump discontinuities. It can also happen at the endpoints of the interval under study. If \( f(-L) \neq f(L) \) then the periodic extension of \( f \) will have a jump discontinuity there. We have the following theorem.

Theorem 145 (Continuity of Fourier Series) Suppose that \( f \) is piecewise smooth on the interval \([-L,L]\). The Fourier series of \( f \) is continuous and converges to \( f \) on \([-L,L]\) if and only if \( f \) is continuous and \( f(-L) = f(L) \).

We are now ready to discuss differentiation and integration of Fourier series.

3.3.4 Differentiation of Fourier Series

Let us start with an example.

Example 146 Consider the Fourier series for \( f(x) = x \) on \([-L,L]\). This Fourier series is (see homework)

\[
x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \frac{2L}{L} \sin \frac{n\pi x}{L}
\]

(3.12)
If we differentiate the left side of equation 3.12 we get \( (x)' = 1 \). The Fourier series for 1 on \([-L, L]\) is (see homework)

\[
1 = 1
\]

However, if we differentiate the right of equation 3.12, we get

\[
2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos \frac{n\pi x}{L},
\]

which is not the Fourier series for 1 on \([-L, L]\) since that series is 1 (see homework).

So, the above example shows us we cannot always differentiate term by term. We now give without proof the conditions under which we can differentiate term by term.

**Theorem 147** If \( f' \) is a piecewise smooth function and if \( f \) is also continuous, then the Fourier series of \( f \) can be differentiated term by term provided that \( f(-L) = f(L) \).

We can see why the example above failed since if \( f(x) = x \) then \( f(-L) \neq f(L) \).

**Remark 148** One advantage of being able to differentiate term by term is to be able to derive new Fourier series from existing ones. This bypasses the lengthy process of computing Euler’ s coefficients. The example below illustrates this.

**Example 149** Find a Fourier series for \( f(x) = -2x \) on \([-1, 1]\) using the theorem above and remembering from the homework of the previous chapter that a Fourier series for \( 1 - x^2 \) on \([-1, 1]\) is

\[
1 - x^2 = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{(n\pi)^2} \cos n\pi x
\]

\( 1 - x^2 \) satisfies the conditions of the theorem, hence we can differentiate its Fourier series term by term. So, we have

\[
\frac{d}{dx} (1 - x^2) = \frac{d}{dx} \left( \frac{2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{(n\pi)^2} \cos n\pi x \right)
\]

or

\[
-2x = \sum_{n=1}^{\infty} \frac{-n\pi (-1)^{n+1} 4}{(n\pi)^2} \sin n\pi x
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^n 4}{n\pi} \sin n\pi x
\]

Figure 149 shows the graph of \(-2x\) along with the graph of \( S_N(x) = \sum_{n=1}^{N} \frac{(-1)^n 4}{n\pi} \sin n\pi x \) for \( N = 5 \) (in red), \( N = 10 \) (in blue) and \( N = 20 \) (in green).
Note that we can even find a Fourier series for \( f(x) = x \) if we divide what we just found by \(-2\). We will obtain
\[
x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin n\pi x
\]
which agrees with what we found in equation 3.12 in the case \( L = 1 \).

### 3.3.5 Integration of Fourier Series

For integration, the situation is much simpler.

**Theorem 150 (Integration of Fourier Series)** A Fourier series of a piecewise smooth function \( f \) can always be integrated term by term and the result is a convergent infinite series that always converges to the integral of \( f \) for \( x \in [-L, L] \).

**Remark 151** The theorem says that a Fourier series can only be integrated term by term and that the result is a convergent infinite series which converges to the integral of \( f \). Note that it does not say it will be a Fourier series. Indeed, it may not be the Fourier series of the function.

**Example 152** The Fourier series for \( 1 - x^2 \) on \([-1, 1]\) is
\[
1 - x^2 = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n\pi)^2} 4 \cos n\pi x
\]
The theorem tells us we can integrate this series term by term. Hence

\[
\int_{-1}^{x} (1 - s^2) \, ds = \int_{-1}^{x} \left[ \frac{2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{(n \pi)^2} \cos n \pi s \right] \, ds
\]

\[
= \int_{-1}^{x} \frac{2}{3} \, ds + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{(n \pi)^2} \int_{-1}^{x} \cos n \pi s \, ds
\]

\[
= \frac{2}{3} (x + 1) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{(n \pi)^2} \left( \frac{1}{n \pi} \sin n \pi s \right)_{-1}^{x}
\]

\[
= \frac{2}{3} (x + 1) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{(n \pi)^2} \frac{1}{n \pi} \sin n \pi x
\]

\[
= \frac{2}{3} (x + 1) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{(n \pi)^2} \sin n \pi x
\]

The result is not a Fourier series because of the term \( \frac{2}{3} (x + 1) \).

### 3.3.6 Problems

1. Show that the Fourier series for \( f(x) = x \) on \([-L, L]\) is
   \[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2L}{n \pi} \sin \frac{n \pi x}{L}.\]

2. Show that the Fourier series for \( f(x) = 1 \) on \([-L, L]\) is 1.

3. Consider the function \( f(x) = x^2 \).
   (a) Find a Fourier series for \( f(x) \) in the interval \([-1, 1]\) by using the theorem on linearity of Fourier series and the known Fourier series for 1 and \( 1 - x^2 \).
   (b) Find a Fourier series for \( g(x) = x \) by differentiating the series you found in part a. First, justify that you can do the differentiation.

4. Consider the function \( f(x) = x \) and its power series which we derived in problem 1.
   (a) Integrate its power series by integrating from \(-L\) to \(x\).
   (b) Is the series we obtained in part a a Fourier series?

5. The general formula for the Fourier series of a piecewise smooth function \( f \) on \([-L, L]\) is
   \[f(x) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n \pi x}{L} + B_n \sin \frac{n \pi x}{L} \right)\]
   (a) Integrate this power series from \(-L\) to \(x\).
   (b) From what you obtain in part a, when will \( \int_{-L}^{x} f(s) \, ds \) be a Fourier series?
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3.3.7 Answers to the Problems

1. Show that the Fourier series for \( f(x) = x \) on \([-L, L]\) is
\[\sum_{n=1}^{\infty} \frac{(-1)^n+1}{n\pi} 2L \sin \frac{n\pi x}{L}.\]

No answer, this is a proof.

2. Show that the Fourier series for \( f(x) = 1 \) on \([-L, L]\) is 1.

No answer, this is a proof.

3. Consider the function \( f(x) = x^2 \).
   (a) Find a Fourier series for \( f(x) \) in the interval \([-1, 1]\) by using the theorem on linearity of Fourier series and the known Fourier series for 1 and \( 1-x^2 \).
   \[x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4}{(n\pi)^2} \cos n\pi x\]
   (b) Find a Fourier series for \( g(x) = x \) by differentiating the series you found in part a. First, justify that you can do the differentiation.
   \[x = \sum_{n=1}^{\infty} \frac{(-1)^n+1}{n\pi} 2L \sin \frac{n\pi x}{L}\]

4. Consider the function \( f(x) = x \) and its power series which we derived in problem 1.
   (a) Integrate its power series by integrating from \(-L\) to \(x\).
   \[\int_{-L}^{x} f(s) \, ds = \sum_{n=1}^{\infty} \frac{(-1)^n 2L^2}{(n\pi)^2} \left[ \cos \frac{n\pi x}{L} - (-1)^n \right]\]

5. The general formula for the Fourier series of a piecewise smooth function \( f \) on \([-L, L]\) is
\[f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L}\right)\]
   (a) Integrate this power series from \(-L\) to \(x\).
   \[\int_{-L}^{x} f(s) \, ds = A_0 (x + L) + \sum_{n=1}^{\infty} \frac{L}{n\pi} \left(A_n \sin \frac{n\pi x}{L} - B_n \left(\cos \frac{n\pi x}{L} - (-1)^n\right)\right)\]
   (b) From what you obtain in part a, when will \( \int_{-L}^{x} f(s) \, ds \) be a Fourier series?
   hint: what is the general form of a Fourier series?