Chapter 7

Isomorphisms

You may remember when we were studying cyclic groups, we made the remark that cyclic groups were similar to $\mathbb{Z}_n$. Indeed, we proved that every cyclic group was Abelian using the fact that addition of integers is Abelian. We expand on this notion. More specifically, we will develop a way to determine if two groups have similar properties. The advantage of this is that if we could tell that two groups $G_1$ and $G_2$ have similar properties and we already know all the properties of $G_1$, then we would immediately know all the properties of $G_2$. The tool which will allow us to do this is called an isomorphism, from the Greek words *isos* which mean "same" and *morphe* which means "form". In this chapter, we will do the following:

1. Define what an isomorphism is.
2. Look at examples of isomorphisms.
3. Prove an isomorphism does what we claim it does (preserves properties).

7.1 Definitions and Elementary Examples

**Definition 272 (Isomorphism)** Let $G$ and $H$ be two groups. We will use multiplication for the notation of their operations, though the operation on $G$ may not be the same as the one on $H$.

1. An **isomorphism** from $G$ to $H$ is a bijection $\phi : G \to H$ with the property that $\phi(ab) = \phi(a)\phi(b)$ for every $a, b$ in $G$. This property means that $\phi$ preserves the group operations.

2. If there exists an isomorphism between $G$ and $H$, we say that $G$ and $H$ are **isomorphic** and we write $G \cong H$. Note that some texts also use $\approx$, even $\sim$.

3. An isomorphism $\phi : G \to G$ is called an **automorphism**, that is an isomorphism of a group to itself.
Let us clarify a few things in this definition.

- $a$ and $b$ are in $G$. When we write $ab$, the operation involved is the operation on $G$.
- $\phi(a)$ and $\phi(b)$ are elements in $H$. When we write $\phi(a)\phi(b)$, the operation involved is the operation on $H$.
- The table below shows the various forms the operation preserving equality $\phi(ab) = \phi(a)\phi(b)$ would take depending on the operations on $G$ and $H$.

<table>
<thead>
<tr>
<th>Operation on $G$</th>
<th>Operation on $H$</th>
<th>Operation preserving</th>
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<tbody>
<tr>
<td>·</td>
<td>·</td>
<td>$\phi(a \cdot b) = \phi(a) \cdot \phi(b)$</td>
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<tr>
<td>·</td>
<td>+</td>
<td>$\phi(a \cdot b) = \phi(a) + \phi(b)$</td>
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<tr>
<td>+</td>
<td>·</td>
<td>$\phi(a + b) = \phi(a) \cdot \phi(b)$</td>
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<tr>
<td>+</td>
<td>+</td>
<td>$\phi(a + b) = \phi(a) + \phi(b)$</td>
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To prove that two groups $G$ and $H$ are isomorphic actually requires four steps, highlighted below:

1. Define a function (mapping) $\phi : G \rightarrow H$ which will be our candidate.
2. Prove $\phi$ is an injection that is $\phi(a) = \phi(b) \implies a = b$.
3. Prove $\phi$ is a surjection that is every element $h$ in $H$ is of the form $h = \phi(g)$ for some $g$ in $G$.
4. Prove $\phi$ preserves the group operations that is $\phi(ab) = \phi(a)\phi(b)$.

**Remark 273** In an isomorphism, there are two parts. The bijection part, is there to ensure that there is a one to one correspondence between the elements of each group. The operation preserving equality ensures that the operation on each group is preserved.

**Remark 274** To prove two groups are not isomorphic, it is not enough to claim one did not find an isomorphism. One has to prove no isomorphism can exist.

We now look at some examples of isomorphisms.

**Example 275** Let $G = (\mathbb{R}, +)$ and $H = (\mathbb{R}^+, .)$. We already know both $G$ and $H$ are groups. We claim $G \cong H$. The function $\phi$ is defined by: $\phi : G \rightarrow H$ such that $\phi(x) = 2^x$. First, we recognize an exponential function, so $\phi$ is a function. It is also an injection. Suppose $\phi(a) = \phi(b)$ then $2^a = 2^b$. Taking the logarithm base 2 on each side gives $a = b$. It is also onto. If $y \in H$, one can find $x \in G$ such that $\phi(x) = y$. For this to happen, we need to have $2^x = y$, solving for $x$ gives $x = \log_2 y$. Finally, if $a$ and $b$ are two elements of $G$, then

$$\phi(a + b) = 2^{a+b} = 2^a2^b = \phi(a)\phi(b)$$
Remark 276 Any exponential function would have worked in the previous example. We had to pick one, so we picked $2^x$ but it could have been $a^x$ where $a \in \mathbb{R}$, $a > 0$ and $a \neq 1$.

Example 277 Any infinite cyclic group is isomorphic to $\mathbb{Z}$. Let $G = \langle a \rangle$ where $|a| = \infty$. Then, $G = \{a^k \mid k \in \mathbb{Z}\}$. Define $\phi : G \to \mathbb{Z}$ by $\phi (a^k) = k$. Then, $\phi$ is a function from $G$ to $\mathbb{Z}$. It is an injection for if $a^i = a^j$ then $i = j$ (see theorem in the previous chapters). It is also onto. Finally

$$\phi (a^k a^j) = \phi (a^{k+j})$$
$$= k + j$$
$$= \phi (a^k) + \phi (a^j)$$

Example 278 Any finite cyclic group $\langle a \rangle$ of order $n$ is isomorphic to $\mathbb{Z}_n$ with the mapping $\phi : \langle a \rangle \to \mathbb{Z}_n$ defined by $\phi (a^k) = k \mod n$. The proof that such a mapping is an isomorphism is left as an exercise.

Example 279 Let $G = \text{SL}(2, \mathbb{R})$ with matrix multiplication. Recall, this is the group of $2 \times 2$ matrices with real entries and determinant equal to $1$. Pick a matrix $M$ from $G$. Define $\phi : G \to G$ by $\phi (A) = MAM^{-1}$. $\phi$ is an automorphism. The proof is left as an exercise.

Example 280 Consider the group $(\mathbb{C}, +)$ and define $\phi : \mathbb{C} \to \mathbb{C}$ by $\phi (a + bi) = a - bi$. $\phi$ is an automorphism. We leave it to the reader to check $\phi$ is a bijection. We verify that it preserves operation. Let $z$ and $w$ be two elements of $\mathbb{C}$ that is $z = a + bi$ and $w = c + di$. Then,

$$\phi (z + w) = \phi (a + c + (b + d) i)$$
$$= a + c - (b + d) i$$
$$= (a - bi) + (c - di)$$
$$= \phi (z) + \phi (w)$$

Example 281 Consider the group $\mathbb{C}^*$ with complex multiplication. The mapping $\phi$ as defined above from $\mathbb{C}^* \to \mathbb{C}^*$ is an automorphism. See problems.

Next, we look at two groups which cannot be isomorphic to illustrate how one might prove no isomorphism can exist between two given groups.

Example 282 $(\mathbb{Q}, +)$ is not isomorphic to $(\mathbb{Q}^*, \cdot)$. As noted earlier, it is not enough to say one did not find an isomorphism to conclude one does not exist. One has to prove an isomorphism cannot exist. If such an isomorphism $\phi$ existed, it would be onto. Since $-1 \in \mathbb{Q}^*$, there would exist $x \in \mathbb{Q}$ such that $\phi (x) = -1$. Writing $x = \frac{x}{2} + \frac{x}{2}$ and using the properties of $\phi$, we see that

$$\phi (x) = -1 \iff \phi \left( \frac{x}{2} + \frac{x}{2} \right) = -1$$
$$\iff \phi \left( \frac{x}{2} \right) \phi \left( \frac{x}{2} \right) = -1$$
$$\iff \left[ \phi \left( \frac{x}{2} \right) \right]^2 = -1$$
which is not possible in the set of rational numbers.

The next example illustrates that an isomorphism is two parts: bijection and operation preserving. Not every bijection is an isomorphism.

**Example 283** This example illustrates the fact that being a bijection is not enough to be an isomorphism. Consider the group \((\mathbb{R},+)\) and the mapping \(\phi : \mathbb{R} \to \mathbb{R}\) defined by \(\phi(x) = x^3\). We leave it to the reader to verify \(\phi\) is a bijection. However, \(\phi\) is not an isomorphism. If it were, we would have \(\phi(x + y) = \phi(x) + \phi(y)\). But \(\phi(x + y) = (x + y)^3\) and \(\phi(x) + \phi(y) = x^3 + y^3\). The two are clearly not equal.

**Example 284** This example illustrates the power of the operation preserving equality \(\phi(ab) = \phi(a)\phi(b)\) and why we call it operation preserving. This equality says that if we have two isomorphic groups \(G\) and \(H\) and we know the operation table of one of them, say \(G\), then we automatically know the operation table of \(H\). Suppose that \(G = U(12) = \{1, 5, 7, 11\}\). Its multiplication table is given by

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>5</th>
<th>7</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>11</td>
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<td>5</td>
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<td>11</td>
<td>11</td>
<td>7</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

Let say that \(H = \{a, b, c, d\}\) and the isomorphism \(\phi : G \to H\) maps the elements as follows: \(\phi(1) = a\), \(\phi(5) = b\), \(\phi(7) = c\) and \(\phi(11) = d\). Suppose we want to know what \(bc\) is. Then, we have

\[
bc = \phi(5) \phi(7) \\
= \phi(5 \cdot 7) \text{ (operation preserving equality)} \\
= \phi(11) \text{ (from the table of } U(12) \text{)} \\
= d
\]

This way, we can build the operation table for \(H\), simply by knowing that of an isomorphic group. It would be:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
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</thead>
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<td>a</td>
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<tr>
<td>d</td>
<td>d</td>
<td>c</td>
<td>b</td>
<td>a</td>
</tr>
</tbody>
</table>

From this example, you can see that the identity of \(H\) is \(a\), it is the image of 1, the identity of \(G\). We will see this is one of the properties of isomorphisms. They map the identity of one group into the identity of the other.

We now prove that an isomorphism as defined above does indeed preserve group properties.
7.2 Properties of Isomorphisms

Instead of giving a different theorem for each result, we will group the results in two main theorems. Since an isomorphism maps the elements of a group into the elements of another group, we will look at the properties of isomorphisms related to their action on elements. Since an isomorphism also acts on all the elements of a group, it acts on the group. We will also look at the properties of isomorphisms related to their action on groups.

**Theorem 285 (Isomorphisms Acting on Group Elements)** Let $G$ and $H$ be two groups. Suppose that $\phi : G \to H$ is an isomorphism. Then the following is true:

1. $\phi$ maps the identity of $G$ into the identity of $H$. In other words, if $e_G$ is the identity of $G$ and $e_H$ is the identity of $H$, then $\phi(e_G) = e_H$.

2. $\phi(a^{-1}) = [\phi(a)]^{-1}$

3. For every integer $n$ and every $a \in G$, $\phi(a^n) = [\phi(a)]^n$

4. For any two elements $a$ and $b$ in $G$, $a$ and $b$ commute if and only if $\phi(a)$ and $\phi(b)$ commute.

5. $G = \langle a \rangle \iff H = \langle \phi(a) \rangle$.

6. Isomorphisms preserve order in other words, for every $a$ in $G$, $|a| = |\phi(a)|$.

7. For a fixed integer $k$ and a fixed group element $b$ in $G$, the equation $x^k = b$ has the same number of solutions in $G$ as does the equation $x^k = \phi(b)$ in $H$.

8. If $G$ is finite then $G$ and $H$ have exactly the same number of elements of every order.

**Proof.** We prove each part separately.

1. Using the notation of the theorem, we have that $e_G = e_G e_G$ hence $\phi(e_G) = \phi(e_G e_G) = \phi(e_G) \phi(e_G)$. But $\phi(e_G)$ is an element of $H$ hence $\phi(e_G) = e_H \phi(e_G)$. Thus, we have $e_H \phi(e_G) = \phi(e_G) \phi(e_G)$. Cancelling $\phi(e_G)$ gives the desired result.

2. Note that $e_G = aa^{-1}$. Hence, from part1, $e_H = \phi(e_G) = \phi(aa^{-1}) = \phi(a) \phi(a^{-1})$. By the uniqueness of inverses, it follows that $\phi(a^{-1}) = [\phi(a)]^{-1}$.

3. We consider 3 cases: $n > 0$, $n = 0$, and $n < 0$. If $n > 0$, we can use mathematical induction. The result is clearly true for $n = 2$. 

\( \phi(a^2) = \phi(aa) = \phi(a) \phi(a) = [\phi(a)]^2 \). Assume the result is true for \( k \), show true for \( k + 1 \).

\[ \phi(a^{k+1}) = \phi(a^k a) = \phi(a^k) \phi(a) = [\phi(a)]^k \phi(a) \text{ by induction hypothesis} \]
\[ = [\phi(a)]^{k+1} \]

Next, if \( n = 0 \), then \( \phi(a^0) = \phi(e_G) = e_H = [\phi(a)]^0 \). Finally, if \( n < 0 \), then \( n = -m \) and \( m > 0 \). Now,

\[ \phi(a^n) = \phi(a^{-m}) = \phi((a^{-1})^m) = [\phi(a^{-1})]^m = \left[ [\phi(a)]^{-1} \right]^m \text{ by part 2} \]
\[ = [\phi(a)]^{-m} = [\phi(a)]^n \]

4. Left as an exercise

5. First, assume \( G = \langle a \rangle \) and show \( H = \langle \phi(a) \rangle \). Note that by closure \( \langle \phi(a) \rangle \subseteq H \). If \( y \in H \) then \( y = \phi(a^k) = [\phi(a)]^k \) hence \( y \in \langle \phi(a) \rangle \) thus \( H \subseteq \langle \phi(a) \rangle \). Thus, the two are equal. Next, suppose \( H = \langle \phi(a) \rangle \) and show \( G = \langle a \rangle \). \( a \in G \) and thus by closure \( \langle a \rangle \subseteq G \). If \( b \in G \) then \( \phi(b) = [\phi(a)]^k = \phi(a^k) \) hence \( b = a^k \) since \( \phi \) is a bijection. This means that \( b \in \langle a \rangle \) and thus \( G \subseteq \langle a \rangle \). It follows that the two are equal.

6. Left as an exercise

7. Left as an exercise

8. Left as an exercise

\[ \] Let us make some further remarks.

**Remark 286** If the operation of \( H \) is addition, then property 3 becomes \( \phi(a^n) = n \phi(a) \). If both the operation of \( G \) and \( H \) are addition, then property 3 becomes \( \phi(na) = n \phi(a) \).

**Remark 287** Property 5 says that an isomorphism maps a generator into a generator.
Remark 288 Property 7 is often used to prove that no isomorphism can exist between two groups. For example, consider the equation $x^4 = 1$ and the groups $\mathbb{R}$ and $\mathbb{C}$ with multiplication. In $\mathbb{R}$ this equation has 2 solutions while in $\mathbb{C}$ it has 4. Hence, no isomorphism can exist between the two.

Before we state the next theorem, let us define some concepts.

Definition 289 (Image and Inverse Image of a Set) Let $f : A \to B$ be a function. Let $E \subseteq A$ and $H \subseteq B$.

1. The image of $G$ under $f$, denoted $f(E)$, is defined to be:
   $$f(E) = \{ f(x) \mid x \in E \}$$

2. The inverse image of $H$ under $f$, denoted $f^{-1}(H)$, is defined to be:
   $$f^{-1}(H) = \{ x \in A \mid f(x) \in H \}$$

Note that this definition does not require $f$ to have an inverse. Though, in our case, since we are dealing with isomorphisms, $f$ will have an inverse.

Theorem 290 (Isomorphisms Acting on Groups) Let $G$ and $H$ be two groups. Suppose that $\phi : G \to H$ is an isomorphism. Then the following is true:

1. $\phi^{-1}$ is an isomorphism from $H$ into $G$.
2. $G$ is Abelian if and only if $H$ is.
3. $G$ is cyclic if and only if $H$ is.
4. If $K$ is a subgroup of $G$ then $\phi(K)$ is a subgroup of $H$.
5. If $J$ is a subgroup of $H$ then $\phi^{-1}(J)$ is a subgroup of $G$.

Proof. We proof each part separately.

1. left as an exercise
2. left as an exercise
3. left as an exercise
4. left as an exercise
4. $\phi(K) \neq \emptyset$, it contains the identity of $H$, $e_H$ from the previous theorem. We can use the test for subgroup. First we check closure under the operation. Let $x$ and $y$ be two elements of $\phi(K)$, show $xy$ is also in $\phi(K)$. There exist $a$ and $b$ in $K$ such that $\phi(a) = x$ and $\phi(b) = y$. Note that since $K$ is itself a group, $ab \in K$. Now, $xy = \phi(a)\phi(b) = \phi(ab) \in \phi(K)$ since $ab \in K$. Finally, we show closure under inverses. Let $x \in \phi(K)$, show $x^{-1} \in \phi(K)$. There exists $a \in K$ such that $\phi(a) = x$. Since $K$ is a group, $a^{-1} \in K$. so $\phi(a^{-1}) = [\phi(a)]^{-1} = x^{-1} \in \phi(K)$.
5. left as an exercise

We finish this chapter with a classic theorem.
7.3 Cayley’s Theorem

The proof of this theorem is not very difficult, and it is a good exercise in group theory as it uses many concepts previously studied. Before we state and prove the theorem, let us establish some preliminary results.

Lemma 291 Let $G$ be a group and pick $g \in G$. Define $T_g : G \rightarrow G$ by $T_g(x) = gx$. In other words, it is left multiplication by $g$. Then, $T_g$ is a permutation on $G$.

Proof. To prove $T_g$ is a permutation on $G$, we must prove it is a bijection.

- **Injection**: Let $x$ and $y$ be elements of $G$ and suppose $T_g(x) = T_g(y)$. We must prove $x = y$.

  $$T_g(x) = T_g(y) \iff gx = gy \iff x = y$$

- **Bijection**: Let $y \in G$. We must prove there exists $x \in G$ such that $y = gx$. But since we are in a group, $y = gx \iff x = g^{-1}y$.

Lemma 292 Let $G$ be a group and let $H = \{T_g \mid g \in G\}$ where $T_g$ is defined as in lemma 291. Then, $H$ is a group with function composition.

Proof. We have to verify that $(H, \circ)$ satisfies the four properties of a group.

- **Closure**: Let $T_a$ and $T_b$ be two elements of $H$. Show $T_a T_b$ belongs to $H$. An element of $H$ is of the form $T_g$ with $g \in G$, so we must prove $T_a T_b = T_g$ for some $g$ in $G$. Since $T_a$ and $T_b$ are two elements of $H$, it means that $a$ and $b$ belong to $G$ hence $ab \in G$ by closure since $G$ is a group. $T_a T_b$ is a permutation on $G$, hence a function, so we see how it acts on elements of $G$. If $x$ is an arbitrary element of $G$, then $T_a T_b(x) = T_a(bx) = abx = T_{ab}(x)$. Since $x$ was arbitrary, we see that $T_a T_b = T_{ab} \in H$ since $ab \in G$.

- **Identity**: $T_e$ where $e$ is the identity of $G$ is the identity of $H$ since $T_a T_e = T_a$ and $T_e T_a = T_e$.

- **Inverses**: Let $T_a \in H$. Then $T_a T_{a^{-1}} = T_a a^{-1} = T_a$ and $T_{a^{-1}} T_a = T_{a^{-1}} a = T_e$. Thus $(T_a)^{-1} = T_{a^{-1}}$.

- **Associativity**: Composition of functions is always associative. You should know this from previous classes, we also proved it in one of the homework problems from the previous chapter.

We are now ready to state and prove our main theorem.
7.3. CAYLEY’S THEOREM

Theorem 293 (Cayley’s Theorem) Every group is isomorphic to a group of permutations.

Proof. Let $G$ be a group. We need to find a group of permutation $G$ is isomorphic to. In particular, we will also need to find the isomorphism. Since $G$ is the only thing we have to start with, we must construct everything else from $G$. For each $g \in G$, we define $T_g$ as in lemma 291. Then, by the same lemma, $T_g$ is a permutation on $G$. We let $H$ be as in lemma 292. by the same lemma, $H$ is a group of permutations. We claim $G \cong H$. We need to produce an isomorphism between the two groups. It is not hard to find. Define $\phi : G \to H$ by $\phi (a) = T_a$. We verify that $\phi$ is an injection, a bijection and operation preserving.

- **Injection**: Let $a, b$ be elements of $G$ and suppose that $\phi (a) = \phi (b)$, we must prove that $a = b$.

  $\phi (a) = \phi (b) \iff T_a = T_b$

  $\iff T_a (e) = T_b (e)$

  $\iff ae = be$

  $\iff a = b$

- **Surjection**: $\phi$ is a surjection by definition since the elements of $H$ are built from the elements of $G$.

- **Operation Preserving**: Let $a, b$ be elements of $G$, we must show that $\phi (ab) = \phi (a) \phi (b)$. From the proof of the lemma, we recall that $T_a T_b = T_{ab}$ hence

  $\phi (ab) = T_{ab}$

  $= T_a T_b$

  $= \phi (a) \phi (b)$

We illustrate this with an example.

Example 294 Consider $U (12) = \{ 1, 5, 7, 11 \}$ with multiplication mod 12. We find the group of permutations it is isomorphic to. Recall from the proof that $U (12)$ is isomorphic to $H = \{ T_g \mid g \in U (12) \} = \{ T_1, T_5, T_7, T_{11} \}$. Let us write the elements of $H$ in array notation.

\[
T_1 = \begin{pmatrix} 1 & 5 & 7 & 11 \\ 1 & 5 & 7 & 11 \end{pmatrix}
\]

\[
T_5 = \begin{pmatrix} 1 & 5 & 7 & 11 \\ 5 & 1 & 11 & 7 \end{pmatrix}
\]

\[
T_7 = \begin{pmatrix} 1 & 5 & 7 & 11 \\ 7 & 11 & 1 & 5 \end{pmatrix}
\]
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\[ T_{11} = \begin{pmatrix} 1 & 5 & 7 & 11 \\ 11 & 7 & 5 & 1 \end{pmatrix} \]

Now, we write both multiplication tables:

\[
\begin{array}{cccc}
U & 1 & 5 & 7 & 11 \\
1 & 1 & 5 & 7 & 11 \\
5 & 5 & 1 & 11 & 7 \\
7 & 7 & 11 & 1 & 5 \\
11 & 11 & 7 & 5 & 1 \\
\end{array}
\]

and

\[
\begin{array}{cccc}
H & T_1 & T_5 & T_7 & T_{11} \\
T_1 & T_1 & T_5 & T_7 & T_{11} \\
T_5 & T_5 & T_1 & T_{11} & T_7 \\
T_7 & T_7 & T_{11} & T_3 & T_5 \\
T_{11} & T_{11} & T_7 & T_5 & T_1 \\
\end{array}
\]

We can see that the tables are identical, except for the notation of the elements. In particular, we can verify \((ab) = (a)(b)\). For example, \(5 \cdot 7 = 11\). Hence on one hand \(\phi(5 \cdot 7) = \phi(11) = T_{11}\). Also, \(\phi(5 \cdot 7) = \phi(5) \phi(7) = T_5 T_7\). We can verify that \(T_5 T_7 = T_{11}\).

7.4 Problems

Do the following problems.

1. Prove that the mapping defined in example 278 is an isomorphism.
2. Prove that the mapping defined in example 283 is a bijection.
3. Prove that the mapping defined in example 279 is an isomorphism.
5. Finish proving theorem 290.
6. Prove that the mapping defined in example 280 is a bijection.
7. Prove that the mapping defined in example 281 is an automorphism.
8. Let \(G\) be the set of all possible groups. Define a relation \(R\) on \(G\) by \(ARB \iff A \cong B\). Prove this is an equivalence relation.
9. Do # 1, 3, 4, 5, 8, 19, 25, 27, 34 at the end of chapter 6.