4.3 Limit of a Sequence: Theorems

These theorems fall in two categories. The first category deals with ways to combine sequences. Like numbers, sequences can be added, multiplied, divided, ... Theorems from this category deal with the ways sequences can be combined and how the limit of the result can be obtained. If a sequence can be written as the combination of several "simpler" sequences, the idea is that it should be easier to find the limit of the "simpler" sequences. These theorems allow us to write a limit in terms of easier limits. however, we still have limits to evaluate. The second category of theorems deal with specific sequences and techniques applied to them. Usually, computing the limit of a sequence involves using theorems from both categories.

4.3.1 Limit Properties

We begin with a few technical theorems. They do not play an important role in computing limits, but they play a role in proving certain results about limits.

**Theorem 4.3.1** Let \( x \) be a number such that \( \forall \epsilon > 0, |x| < \epsilon \), then \( x = 0 \).

**Proof.** See problems at the end of the section.

**Theorem 4.3.2** If a sequence converges, then its limit is unique.

**Proof.** We assume that \( a_n \to L_1 \) and \( a_n \to L_2 \) and show that \( L_1 = L_2 \). Given \( \epsilon > 0 \) choose \( N_1 \) such that \( n \geq N_1 \implies |a_n - L_1| < \frac{\epsilon}{2} \). Similarly, choose \( N_2 \) such that \( n \geq N_2 \implies |a_n - L_2| < \frac{\epsilon}{2} \). Let \( N = \max (N_1, N_2) \). If \( n \geq N \), then

\[
|L_1 - L_2| = |L_1 - a_n + a_n - L_2| \\
\leq |a_n - L_1| + |a_n - L_2| \\
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

By theorem 4.3.1, it follows that \( L_1 - L_2 = 0 \), that is \( L_1 = L_2 \).

**Theorem 4.3.3** If a sequence converges, then it is bounded, that is there exists a number \( M > 0 \) such that \( |a_n| \leq M \) for all \( n \).

**Proof.** Choose \( N \) such that \( n \geq N \implies |a_n - L| < 1 \). By the triangle inequality, we have

\[
|a_n| - |L| \leq |a_n - L| \\
< 1
\]

Thus, if \( n \geq N \), \( |a_n| < 1 + |L| \). Let \( M_1 = \max (|a_1|, |a_2|, ..., |a_{N-1}|) \). Let \( M = \max (M_1, 1 + |L|) \). Then, clearly, \( |a_n| < M \).

**Theorem 4.3.4** If a sequence converges, then \( \forall \epsilon > 0 \exists N : m, n \geq N \implies |a_n - a_m| < \epsilon \).
**Proof.** Given $\epsilon > 0$, we can choose $N$ such that $n, m \geq N \implies |a_n - L| < \frac{\epsilon}{2}$ and $|a_m - L| < \frac{\epsilon}{2}$. Now,

\[
|a_n - a_m| = |a_n - L + L - a_m| \\
= |(a_n - L) - (a_m - L)| \\
\leq |a_n - L| + |a_m - L| \text{ by the triangle inequality} \\
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

The above theorem simply says that if a sequence converges, then the difference between any two terms gets smaller and smaller. It should also be clear to the reader that if $a_n \to L$, then so does $a_{n+k}$ where $k$ is any natural number.

The above theorem can be used to prove that a sequence does not converge by proving that the difference between two of its terms does not get smaller and smaller.

**Example 4.3.5** Find $\lim \cos n\pi$

We suspect the sequence diverges, as its values will oscillate between -1 and 1. We can actually prove it using theorem 4.3.4. We notice that for any $n$,

\[
|\cos 2n\pi - \cos ((2n + 1)\pi)| = 2.
\]

For the sequence to converge, this difference should approach 0. Hence, the sequence diverges.

**Theorem 4.3.6** Suppose that $(a_n)$ converges. Then, any subsequence $(a_{nk})$ also converges and has the same limit.

**Proof.** Suppose that $a_n \to L$. Let $b_k = a_{nk}$ be a subsequence. We need to prove that given $\epsilon > 0$, there exists $N$ such that $k \geq N \implies |b_k - L| < \epsilon$. Let $\epsilon > 0$ be given. Choose $N$ such that $n_k \geq N \implies |a_{n_k} - L| < \epsilon$. Now, if $k \geq N$, then $n_k \geq N$ therefore

\[
|b_k - L| = |a_{n_k} - L| \\
< \epsilon
\]

This theorem is often used to show that a given sequence diverges. To do so, it is enough to find two subsequences which do not converge to the same limit. Alternatively, once can find a subsequence which diverges.

**Example 4.3.7** Study the convergence of $\cos n\pi$

The subsequence $\cos 2n\pi$ converges to 1, while the subsequence $\cos (2n + 1)\pi$ converges to -1. Thus, $\cos 2n\pi$ must diverge.

In the next two sections, we look at theorems which give us more tools to compute limits.
4.3. LIMIT OF A SEQUENCE: THEOREMS

4.3.2 Limit Laws

The theorems below are useful when finding the limit of a sequence. Finding the limit using the definition is a long process which we will try to avoid whenever possible. Since all limits are taken as $n \to \infty$, in the theorems below, we will write $\lim_{n \to \infty} a_n$ for $\lim_{n \to \infty} a_n$.

**Theorem 4.3.8** Let $(a_n)$ and $(b_n)$ be two sequences such that $a_n \to a$ and $b_n \to b$ with $a$ and $b$ real numbers. Then, the following results hold:

1. $\lim (a_n \pm b_n) = (\lim a_n) \pm (\lim b_n) = a \pm b$.
2. $\lim (a_nb_n) = (\lim a_n)(\lim b_n) = ab$.
3. If $\lim b_n = b \neq 0$ then $\lim \left( \frac{a_n}{b_n} \right) = \frac{\lim a_n}{\lim b_n} = \frac{a}{b}$.
4. $\lim |a_n| = |\lim a_n| = |a|$.
5. If $a_n \geq 0$ then $\lim a_n \geq 0$.
6. If $a_n \geq b_n$ then $\lim a_n \geq \lim b_n$.
7. If $\lim a_n = a \geq 0$ then $\lim \sqrt{a_n} = \sqrt{\lim a_n} = \sqrt{a}$.

**Proof.** We prove some of these items. The remaining ones will be assigned as problems at the end of the section.

1. We prove $\lim (a_n + b_n) = (\lim a_n) + (\lim b_n)$. The proof of $\lim (a_n - b_n) = (\lim a_n) - (\lim b_n)$ is left as an exercise. We need to prove that $\forall \varepsilon > 0$, $\exists N : n \geq N \implies |a_n + b_n - (a + b)| < \varepsilon$. Let $\varepsilon > 0$ be given, choose $N_1$ such that $n \geq N_1 \implies |a_n - a| < \frac{\varepsilon}{2}$. Choose $N_2$ such that $n \geq N_2 \implies |b_n - b| < \frac{\varepsilon}{2}$. Let $N = \max (N_1, N_2)$. If $n \geq N$, then

\[|a_n + b_n - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|\] by the triangle inequality

\[< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\]

2. We need to prove that $\forall \varepsilon > 0$, $\exists N : n \geq N \implies |a_nb_n - ab| < \varepsilon$. Since $a_n$ converges, it is bounded, let $M$ be the bound i.e. $|a_n| < M$. Choose $N_1$ such that $n \geq N_1 \implies |a_n - a| < \frac{\varepsilon}{2(|b| + 1)}$. Choose $N_2$ such that
CHAPTER 4. SEQUENCES AND LIMIT OF SEQUENCES

\[ n \geq N_2 \implies |b_n - b| < \frac{\epsilon}{2(M + 1)}. \] Let \( N = \max(N_1, N_2). \) If \( n \geq N \) then

\[ |a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| \]

\[ = |a_n (b_n - b) + b (a_n - a)| \]

\[ \leq |a_n| |b_n - b| + |b| |a_n - a| \]

\[ < \frac{M}{2(M + 1)} + \frac{\epsilon}{2(|b| + 1)} \]

\[ < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \]

3. See problems

4. We need to prove that \( |a_n| \to |a| \) that is \( \forall \epsilon > 0, \exists N : n \geq N \implies ||a_n| - |a|| < \epsilon. \) Let \( \epsilon > 0 \) be given, choose \( N \) such that \( n \geq N \implies |a_n - a| < \epsilon \) (since \( a_n \to a \)). If \( n \geq N, \) then we have:

\[ ||a_n| - |a|| < |a_n - a| \text{ by the triangle inequality} \]

\[ < \epsilon \]

5. We prove it by contradiction. Assume that \( a_n \to a < 0. \) Choose \( N \) such that \( n \geq N \implies |a_n - a| < -\frac{1}{2}a. \) Then,

\[ a_n - a < -\frac{1}{2}a \]

\[ \implies a_n < \frac{1}{2}a \]

\[ \implies a_n < 0 \]

which is a contradiction.

6. We apply the results found in parts 1 and 5 to the sequence \( a_n - b_n. \)

7. See problems

- Remark 4.3.9 Parts 1, 2 and 3 of the above theorem hold even when \( a \) and \( b \) are extended real numbers as long as the right hand side in each part is defined. You will recall the following rules when working with extended real numbers:

1. \( \infty + \infty = \infty \times \infty = (-\infty)(-\infty) = \infty \)
2. \( -\infty - \infty = (-\infty)\infty = \infty(-\infty) = -\infty \)
3. If \( x \) is any real number, then

\[ (a) \infty + x = x + \infty = \infty \]
4.3. LIMIT OF A SEQUENCE: THEOREMS

(b) \(-\infty + x = x - \infty = -\infty\)

(c) \(\frac{x}{\infty} = \frac{x}{-\infty} = 0\)

(d) \(\frac{x}{0} = \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases}\)

(e) \(\infty \times x = x \times \infty = \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases}\)

(f) \((-\infty) \times x = x \times (-\infty) = \begin{cases} -\infty & \text{if } x > 0 \\ \infty & \text{if } x < 0 \end{cases}\)

4. However, the following are still indeterminate forms. Their behavior is unpredictable. Finding what they are equal to requires more advanced techniques such as l’Hôpital’s rule.

(a) \(-\infty + \infty \text{ and } \infty - \infty\)

(b) \(0 \times \infty \text{ and } \infty \times 0\)

(c) \(\frac{\infty}{\infty} \text{ and } \frac{0}{0}\)

(d) \(0^0, \infty^0, 0^\infty, \text{ and } 1^\infty\)

**Remark 4.3.10** When using theorems from this category, it is important to remember previous results since these theorems allow us to write a limit in terms of other limits, we hopefully know. The more limits we know, the better off we are.

**Example 4.3.11** If \(c \neq 0\), find \(\lim_{n \to \infty} \frac{c}{n}\). We know from an example in the previous section that \(\lim_{n \to \infty} \frac{1}{n} = 0\). Therefore

\[
\lim_{n \to \infty} \frac{c}{n} = c \lim_{n \to \infty} \frac{1}{n} = c \times 0 = 0
\]

**Example 4.3.12** Find \(\lim_{n \to \infty} \frac{1}{n^2}\). In the previous section, we computed this limit using the definition. We can also do it as follows.

\[
\lim_{n \to \infty} \frac{1}{n^2} = \lim_{n \to \infty} \left(\frac{1}{n} \times \frac{1}{n}\right) = \left(\lim_{n \to \infty} \frac{1}{n}\right) \left(\lim_{n \to \infty} \frac{1}{n}\right) = 0 \times 0 = 0
\]

**Remark 4.3.13** From the above example, we can see that if \(p\) is a natural number, \(\lim_{n \to \infty} \frac{1}{n^p} = 0\).
Example 4.3.14 Find \( \lim_{n \to \infty} \frac{n^2 + 3n}{2n^2 + 1} \).

This problem involves using a standard technique you should remember. We show all the steps, then we will draw a general conclusion. We begin by factoring the term of highest degree from both the numerator and denominator.

\[
\lim_{n \to \infty} \frac{n^2 + 3n}{2n^2 + 1} = \lim_{n \to \infty} \frac{n^2 (1 + \frac{3}{n})}{n^2 \left(2 + \frac{1}{n^2}\right)} = \lim_{n \to \infty} \frac{1 + \frac{3}{n}}{2 + \frac{1}{n^2}}
\]

Now,

\[
\lim \left(1 + \frac{3}{n}\right) = \lim (1) + 3 \lim \frac{1}{n} = 1
\]

and

\[
\lim \left(2 + \frac{1}{n^2}\right) = \lim (2) + \lim \frac{1}{n^2} = 2
\]

Since both the limit of the numerator and denominator exist and the limit of the denominator is not 0, we can write

\[
\lim_{n \to \infty} \frac{n^2 + 3n}{2n^2 + 1} = \frac{\lim \left(1 + \frac{3}{n}\right)}{\lim \left(2 + \frac{1}{n^2}\right)} = \frac{1}{2}
\]

Remark 4.3.15 The same technique can be applied to every fraction for which the numerator and denominator are polynomials in \( n \). We see that the limit of such a fraction will be the same as the limit of the quotient of the terms of highest degree. Let us look at some examples:

Example 4.3.16 \( \lim_{n \to \infty} \frac{3n^2 + 2n - 10}{2n + 5} = \lim_{n \to \infty} \frac{3n^2}{2n} = \lim_{n \to \infty} \frac{3}{2} n = \frac{2}{3} \lim n = \infty \)

Example 4.3.17 \( \lim_{n \to \infty} \frac{5n^3 - 2n + 1}{2n^4 + 5n^2 - 2} = \lim_{n \to \infty} \frac{5n^3}{2n^4} = \lim_{n \to \infty} \frac{5}{2} n = \frac{5}{2} \lim \frac{1}{n} = 0 \)

Example 4.3.18 \( \lim_{n \to \infty} \frac{2n^3 - n^2 + 2n + 1}{n^3 + 10n^2 - 5} = \lim_{n \to \infty} \frac{2n^3}{n^3} = \lim 2 = 2 \)
4.3. LIMIT OF A SEQUENCE: THEOREMS

4.3.3 More Theorems on Limits

In example 4.2.18, we used an approximation to simplify the problem a little bit. In this particular example, the approximation was not really necessary, it was more to illustrate a point. Sometimes, if the problem is more complicated, it may be necessary to use such an approximation in order to be able to find the condition \( n \) has to satisfy. In other words, when we try to satisfy \( |x_n - L| < \epsilon \), we usually simplify \( |x_n - L| \) to some expression involving \( n \). Let \( E_1(n) \) denote this expression. This gives us the inequality \( E_1(n) < \epsilon \) which we have to solve for \( n \). If it is too hard, we then try to find a second expression we will call \( E_2(n) \) such that \( E_1(n) < E_2(n) < \epsilon \). \( E_2(n) \) should be such that solving the inequality \( E_2(n) < \epsilon \) is feasible and easier. In order to achieve this, several tricks are used. We recall some useful results, as well as some theorems below.

**Example 4.3.19** Find \( \lim \frac{2n^2 + 1}{n^2 + 1} \) using the definition.

Of course, we can find this limit by using the theorems on limits. Here, we do it using the definition, as asked. We think the limit is 2. We want to show that for every \( \epsilon > 0 \), there exists \( N \) such that \( n \geq N \implies \left| \frac{2n^2 + 1}{n^2 + 1} - 2 \right| < \epsilon \). First, we simplify the absolute value.

\[
\left| \frac{2n^2 + 1}{n^2 + 1} - 2 \right| = \left| \frac{-1}{n^2 + 1} \right| = \frac{1}{n^2 + 1} < \frac{1}{n^2} < \frac{1}{n} \text{ if } n > 1
\]

So, we see that if \( \frac{1}{n} < \epsilon \), which happens when \( n > \frac{1}{\epsilon} \), then we will have \( \left| \frac{2n^2 + 1}{n^2 + 1} - 2 \right| < \epsilon \). So, \( N = \frac{1}{\epsilon} \) will work.

**Example 4.3.20** Find \( \lim \frac{1}{n^2 + 2n - 4} \)

We think the limit is 0. We need to prove that for every \( \epsilon > 0 \), there exists \( N \) such that \( n \geq N \implies \left| \frac{1}{n^2 + 2n - 4} \right| < \epsilon \). We begin by noticing that

\[
n^2 + 2n - 4 = n^2 + 2(n - 1) > n^2 \text{ if } n > 2
\]
Therefore, if \( n > 2 \),
\[
\left| \frac{1}{n^2 + 2n - 4} \right| = \frac{1}{n^2 + 2n - 4} < \frac{1}{n^2} < \frac{1}{n}
\]
If \( n > \max \left( 2, \frac{1}{\epsilon} \right) \), then \( \left| \frac{1}{n^2 + 2n - 4} \right| < \epsilon \). So, \( N = \max \left( 2, \frac{1}{\epsilon} \right) \) will work.

Remark 4.3.21 The two examples above could have been done using the techniques discussed in the previous subsection, that is without using the definition.

We first recall an inequality we proved in the section on proof by induction.

Theorem 4.3.22 (Bernoulli’s inequality) If \( x \geq -1 \), and \( n \) is a natural number, then \( (1 + x)^n \geq 1 + nx \).

We will use this inequality in one of the proofs below.

Theorem 4.3.23 (squeeze theorem) If \( a_n \to L \), \( c_n \to L \) and \( a_n \leq b_n \leq c_n \), then \( b_n \to L \).

Proof. We need to prove that \( \forall \epsilon > 0 \exists N : n \geq N \implies |b_n - L| < \epsilon \). Let \( \epsilon > 0 \) be given. Choose \( N_1 \) such that \( n \geq N_1 \implies |a_n - L| < \epsilon \) or \( -\epsilon < a_n - L < \epsilon \). Similarly, choose \( N_2 \) such that \( n \geq N_2 \implies -\epsilon < c_n - L < \epsilon \). Let \( N = \max (N_1, N_2) \). If \( n \geq N \) then
\[
\begin{align*}
& a_n \leq b_n \leq c_n \\
& \iff a_n - L \leq b_n - L \leq c_n - L \\
& \iff -\epsilon < a_n - L \leq b_n - L \leq c_n - L < \epsilon \\
& \iff -\epsilon < b_n - L < \epsilon \\
& \iff |b_n - L| < \epsilon
\end{align*}
\]

Theorem 4.3.24 If \( 0 < a < 1 \) then \( a^n \to 0 \)

Proof. Let \( x = \frac{1}{a} - 1 \). Then, \( x > 0 \) and \( a = \frac{1}{1 + x} \). For \( n \geq 1 \),
\[
a^n = \frac{1}{(1 + x)^n} \\
\leq \frac{1}{1 + nx} \text{ by Bernoulli’s inequality} \\
< \frac{1}{nx}
\]
To show that \( a^n \to 0 \), we need to show that \( |a^n| < \epsilon \), for any \( \epsilon \) whenever \( n \geq N \), that is

\[
a^n < \epsilon \iff \frac{1}{nx} < \epsilon \\
\iff n > \frac{1}{x\epsilon}
\]

So, given \( \epsilon > 0 \), \( N = \frac{1}{x\epsilon} \) will work. ■

We look at a few more examples, to see how all these results come into play.

**Example 4.3.25** Prove that \( \lim \frac{4^n}{n!} = 0 \)

The trick we use is worth remembering. We will use the squeeze theorem. We note that

\[
0 \leq \frac{4^n}{n!} = \frac{4 \times 4 \times 4 \times \ldots \times 4}{1 \times 2 \times 3 \times \ldots \times n} \\
\leq \frac{4^3}{3!} \times \frac{4}{n} \quad \text{since} \quad \frac{4 \times 4 \times \ldots \times 4}{4 \times 5 \times \ldots \times n - 1} \leq 1
\]

But \( \lim \frac{4^3}{3!} \times \frac{4}{n} = 0 \) hence the result follows by the squeeze theorem.

**Example 4.3.26** Find \( \lim \frac{3^n}{4^n+2} \).

\[
\frac{3^n}{4^n+2} = \frac{3^n}{4^2 \cdot 4^n} = \frac{1}{16} \left( \frac{3}{4} \right)^n
\]

Hence, by the theorem above

\[
\lim \frac{3^n}{4^n+2} = \frac{1}{16} \lim \left( \frac{3}{4} \right)^n = 0 \quad \text{since} \quad 0 < \frac{3}{4} < 1
\]

We now extend Bernoulli’s inequality.

**Theorem 4.3.27** (binomial theorem) \( (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots + nab^{n-1} + b^n \).

**Corollary 4.3.28** \( (1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + nx^{n-1} + x^n \).
In particular, when $x \geq 0$, then $(1 + x)^n$ is greater than any part of the right hand side. For example, we obtain Bernoulli’s inequality: $(1 + x)^n \geq 1 + nx$. We could also write $(1 + x)^n \geq \frac{n(n-1)}{2}x^2$ or $(1 + x)^n \geq \frac{n(n-1)(n-2)}{3!}x^3$. And so on. This is useful to get approximations on quantities like $3^n$. We rewrite it as

$$3^n = (1 + 2)^n$$

**Example 4.3.29** Find $\lim_{n \to \infty} \frac{n}{4^n}$

We suspect the limit will be 0 since the denominator grows much faster than the numerator. So, we need to find a fraction slightly larger than $\frac{n}{4^n}$ which we know converges to 0. Then, we will be able to invoke the squeeze theorem. Using the corollary of the binomial theorem, we note that $4^n = (1 + 3)^n \geq \frac{9n(n-1)}{2}$

thus $0 \leq \frac{n}{4^n} \leq \frac{2n}{9n(n-1)} = \frac{2}{9n-9} \to 0$. Hence, by the squeeze theorem, $\lim_{n \to \infty} \frac{n}{4^n} = 0$.

**Example 4.3.30** Find $\lim_{n \to \infty} \frac{n^2}{4^n}$

This is similar to the problem above. However, the numerator is a higher power of $n$, so we need to write $4^n$ in terms of a higher power of $n$.

$$4^n \geq (1 + 3)^n \geq \frac{3^3n(n-1)(n-2)}{3!} \geq \frac{9n(n-1)(n-2)}{2}$$

Thus,

$$0 \leq \frac{n^2}{4^n} \leq \frac{2n^2}{9n(n-1)(n-2)} \leq \frac{2n}{9(n-1)(n-2)}$$

And $\lim_{n \to \infty} \frac{2n}{9(n-1)(n-2)} = 0$ (why?) hence by the squeeze theorem, $\lim_{n \to \infty} \frac{n}{4^n} = 0$.

**Example 4.3.31** Find $\lim_{n \to \infty} \frac{4^n}{2n}$

We suspect this sequence diverges to infinity. We need to prove that $\frac{4^n}{2n}$ grows without bounds, that is for every $M > 0$, there exists $N$ such that $n \geq N \implies$
\[
\frac{4^n}{2n} > M. \text{ First, we notice that} \\
\frac{4^n}{2n} = \frac{(1 + 3)^n}{2n} > \frac{2}{2n} \quad \text{by using the binomial theorem} \\
> \frac{9(n-1)}{4}
\]

We can see that
\[
\frac{9(n-1)}{4} > M \iff n - 1 > \frac{4M}{9} \\
\iff n > \frac{4M + 9}{9}
\]

so, \(N = \frac{4M + 9}{9}\) will work.

**Remark 4.3.32** We used the binomial theorem to deduce that \((1 + 3)^n \geq \frac{n(n-1)3^2}{2}\).

Often, when using this theorem, students do not know which term to keep, the term in \(x\), or \(x^2\), or \(x^3\),... The answer is that it depends on what we are trying to achieve. Here, we wanted to show that \(\frac{4^n}{2n} > M\) for any \(M\), in other words, \(\frac{4^n}{2n}\) can be made as big as we want simply by taking \(n\) large enough. Since we could not solve for \(n\), we replaced \(\frac{4^n}{2n}\) by a smaller term. That smaller term should have similar properties, in other words, the smaller term should also get arbitrarily large. If we take the smaller term too small, it will not work. For example, it is true that \(\frac{4^n}{2n} > \frac{1}{2n}\). However, \(\frac{1}{2n}\) cannot be made as large as we want, so this does not help us. If instead of using \((1 + 3)^n \geq \frac{n(n-1)3^2}{2}\) (terms of second degree), we had used \((1 + 3)^n \geq 1 + 3n\), then we would have obtained
\[
\frac{1 + 3n}{2n} > M \iff 2nM < 1 + 3n \\
\iff 2nM - 3n < 1 \\
\iff n(2M - 3) < 1 \\
\iff n < \frac{1}{2M - 3}
\]

So, this does not work. This tells us that \(\frac{1 + 3n}{2n} > M\) only for a few values of \(n\). This is why we used terms of higher degree.
Example 4.3.33 Find $\lim (\sqrt{n + 2} - \sqrt{n})$

\[
\lim (\sqrt{n + 2} - \sqrt{n}) = \lim \frac{(\sqrt{n + 2} - \sqrt{n})(\sqrt{n + 2} + \sqrt{n})}{(\sqrt{n + 2} + \sqrt{n})} = \lim \frac{2}{\sqrt{n + 2} + \sqrt{n}} = \lim (\sqrt{n + 2} + \sqrt{n}) = 0
\]

Remark 4.3.34 In many proofs or problems, different versions of the triangle inequality are often used. As a reminder, here are the different versions of the triangle inequality students should remember.

\[||a| - |b|| \leq |a - b| \leq |a| + |b|\]

and

\[||a| - |b|| \leq |a + b| \leq |a| + |b|\]

Finally, we give a theorem which generalizes some of the examples we did above.

Theorem 4.3.35 All limits are taken as $n \to \infty$.

1. If $p > 0$ then $\lim \frac{1}{np} = 0$.
2. If $p > 0$ then $\lim \sqrt[p]{p} = 1$.
3. $\lim \sqrt[3]{n} = 1$.
4. If $p > 1$ and $\alpha \in \mathbb{R}$ then $\lim \frac{n^\alpha}{pn} = 0$.
5. If $|p| < 1$ then $\lim p^n = 0$
6. $\forall p \in \mathbb{R}$, $\lim \frac{p^n}{n!} = 0$.

Proof. We prove each part separately.

1. We can use the definition. Let $\epsilon > 0$ be given. We show there exists $N > 0 : n \geq N \implies \left| \frac{1}{np} - 0 \right| < \epsilon$. We begin with what we want to achieve.

\[
\left| \frac{1}{np} - 0 \right| < \epsilon \iff \frac{1}{np} < \epsilon \\
\iff np > \frac{1}{\epsilon} \iff n > \frac{1}{\sqrt[1/p]{\epsilon}}
\]
So, we see that if \( N \) is the smallest integer larger than \( \frac{1}{\sqrt{n}} \), the result will follow.

2. See problems

3. Let \( x_n = \sqrt[n]{n} - 1 \). Proving the result amounts to proving that \( \lim x_n = 0 \).

From \( x_n = \sqrt[n]{n} - 1 \), we can write

\[
\sqrt[n]{n} = x_n + 1
\]

\[
n = (x_n + 1)^n
\]

Using the binomial theorem, we see that if \( n \geq 2 \)

\[
n \geq \frac{n(n-1)}{2} x_n^2
\]

Thus

\[
x_n^2 \leq \frac{2}{n-1}
\]

It follows that for all \( n \geq 2 \) we have

\[
0 \leq x_n \leq \sqrt{\frac{2}{n-1}}
\]

Since \( \lim \sqrt{\frac{2}{n-1}} = 0 \), it follows using the squeeze theorem that \( \lim x_n = 0 \).

4. Here again, we will use the binomial theorem. Since \( p > 1 \), we can write

\[
p = 1 + q \text{ with } q > 0
\]

Therefore, if \( k \) is a positive integer such that \( k > \alpha \), we have

\[
p^n = (1 + q)^n > \frac{n(n-1)...(n-k+1)q^k}{k!}
\]

Now, if \( n > 2k \), then \( k < \frac{1}{2} n \). It follows that \( n-k+1 > \frac{1}{2} n +1 > \frac{1}{2} n \). It follows that

\[
\frac{n(n-1)...(n-k+1)}{k!} > \frac{n^k}{2^k k!}
\]

and therefore

\[
0 \leq \frac{n^\alpha}{p^n} \leq \frac{2^k k!}{q^k} \times \frac{1}{n^k}
\]

Since

\[
\lim_{n \to \infty} \frac{2^k k!}{q^k} \times \frac{1}{n^k} = \frac{2^k k!}{q^k} \lim_{n \to \infty} \frac{1}{n^k} = 0
\]

It follows from the squeeze theorem that \( \lim \frac{n^\alpha}{p^n} = 0 \)
5. See problems (hint: write \( p = \pm \frac{1}{q} \) and use part 4 of the theorem with \( \alpha = 0 \)).

6. See problems.

4.3.4 Exercises

1. Prove that \( \frac{5^n}{n!} \to 0 \).

2. Prove that \( \frac{n!}{n^n} \to 0 \).

3. Finish proving theorem 4.3.35.

4. Consider \( (x_n) \) a sequence of real numbers such that \( \lim_{n \to \infty} x_n = x \) where \( x > 0 \), prove there exists an integer \( N > 0 \) such that \( n \geq N \implies x_n > 0 \).

5. Consider \( (x_n) \) a sequence of real numbers such that \( x_n \geq 0 \) for any \( n \). If \( x \) is a partial limit of \( (x_n) \), prove that \( x \geq 0 \). Prove the same result if \( x = \lim x_n \). Use this to show that if \( x_n \leq y_n \) then \( \lim x_n \leq \lim y_n \).

6. Consider \( (x_n) \) a sequence of real numbers and let \( x \in \mathbb{R} \). Prove the two conditions below are equivalent.

(a) \( x_n \to x \) as \( n \to \infty \).
(b) \( |x_n - x| \to 0 \) as \( n \to \infty \).

7. Consider \( (x_n) \) a sequence of real numbers and let \( x \in \mathbb{R} \). If \( \lim x_n = x \), prove that \( \lim |x_n| = |x| \).

8. Show by examples that if \( |x_n| \to |x| \) then \( x_n \) does not necessarily converge. If it does converge, it does not necessarily converge to \( x \).

9. Consider \( (x_n) \) a sequence of real numbers and let \( x \in \mathbb{R} \). Suppose that \( \lim x_n = x \). Define \( y_n = x_{n+p} \) for some integer \( p \). Prove that \( \lim y_n = x \).

10. Given that \( a_n \leq b_n \) for every \( n \) and that \( \lim a_n = \infty \), prove that \( \lim b_n = \infty \).

11. Suppose that \( (a_n) \) and \( (b_n) \) are sequences of real numbers such that \( |a_n - b_n| < 1 \) for every \( n \). Prove that if \( \infty \) is a partial limit of \( (a_n) \) then \( \infty \) is also a partial limit of \( (b_n) \).

12. Two sequences are said to be eventually close if \( \forall \epsilon > 0, \exists N > 0 : n \geq N \implies |a_n - b_n| < \epsilon \).
4.3. LIMIT OF A SEQUENCE: THEOREMS

(a) Prove that if two sequences \((a_n)\) and \((b_n)\) are eventually close and if a number \(x\) is the limit of the sequence \((a_n)\) then \(x\) is also the limit of the sequence \((b_n)\).

(b) Prove that if two sequences \((a_n)\) and \((b_n)\) are eventually close and if a number \(x\) is a partial limit of the sequence \((a_n)\) then \(x\) is also a partial limit of the sequence \((b_n)\).

13. Determine if the sequences below have limits (finite or infinite). In each case, if a limit (finite or infinite) exists, prove it using the definition of the limit of a sequence.

(a) \(\frac{n - 1}{n + 1}\)

(b) \(\frac{3n^2 + 1}{n^2 + 1}\)

(c) \(\frac{5n^3 + n - 4}{2n^3 + 3}\)

(d) \(\frac{n^3}{n + 1}\)

(e) \(\frac{4n^2 + 1}{n^3 + n}\)

(f) \(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}\)

(g) \((-1)^{2n-1} \left(1 - \frac{1}{2^n}\right)\)

(h) \(\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \cdots \left(1 - \frac{1}{n}\right)\)

(i) \(\frac{5^n}{2n}\)

14. Prove that if \(a > 1\) then \(a^2 > a > \sqrt{a} > 1\)

15. Prove that \(\frac{n}{2^n} \to 0\), \(\frac{n^2}{2^n} \to 0\), \(\frac{n^3}{2^n} \to 0\), \(\frac{n^4}{2^n} \to 0\)

16. Prove theorem 4.3.1

17. Finish proving theorem 4.3.8

18. Let \((x_n)\) and \((y_n)\) be two sequences. Assume further that \(\lim x_n = x\) and \(y\) is a partial limit of \((y_n)\). Prove that \(x + y\) is a partial limit for \((x_n + y_n)\).

19. State and prove similar results for subtraction, multiplication and division.

20. Give an example of two divergent sequences \((x_n)\) and \((y_n)\) such that \((x_n + y_n)\) is convergent.
21. Give an example of two sequences \((x_n)\) and \((y_n)\) such that \(x_n \to 0, y_n \to \infty\) and:

- (a) \(x_n y_n \to 0\)
- (b) \(x_n y_n \to c\) where \(0 < c < \infty\)
- (c) \(x_n y_n \to \infty\)
- (d) \((x_n y_n)\) is bounded but has no limit.

22. Let \((x_n)\) be a sequence of real numbers such that \(x_n \to 0\). Prove that

\[
\frac{x_1 + x_2 + \ldots + x_n}{n} \to 0
\]

23. Let \((x_n)\) be a sequence of real numbers such that \(x_n \to x\) for some real number \(x\). Prove that

\[
\frac{x_1 + x_2 + \ldots + x_n}{n} \to x
\]

24. Prove that \(|x_n| \to 0 \iff x_n \to 0\).

25. Show by examples that if \((x_n)\) and \((y_n)\) are divergent sequences then \((x_n + y_n)\) is not necessarily divergent. Do the same for \((x_n y_n)\).

26. Show that if \((x_n) \to 0\) and \((y_n)\) is bounded then \((x_n y_n) \to 0\).

27. Prove that if \((x_n)\) is a sequence in a set \(S \subseteq \mathbb{R}\), then every finite partial limit of \((x_n)\) must belong to \(\overline{S}\).

28. Prove that if \(S \subseteq \mathbb{R}\) and \(x \in \overline{S}\) then there exists a sequence \((x_n)\) in \(S\) such that \(x_n \to x\).

29. Let \(S \subseteq \mathbb{R}\). Prove that the following conditions are equivalent:

- (a) \(S\) is unbounded above.
- (b) There exists a sequence \((x_n)\) in \(S\) such that \(x_n \to \infty\).
4.3. LIMIT OF A SEQUENCE: THEOREMS

4.3.5 Hints for the Exercises

1. Prove that \( \frac{5^n}{n!} \to 0 \).
   Hint: see similar example in the notes.

2. Prove that \( \frac{n!}{n^n} \to 0 \).
   Hint: see similar example in the notes.

3. Finish proving theorem 4.3.35.
   Hint: some parts are proven in the notes, but there are several parts left to prove.
   
   - **part 2:** let \( x_n = \sqrt{n} - 1 \) and show that \( \lim x_n = 0 \).
   - **part 5:** use the suggested hint in the notes.
   - **part 6:** Fix \( k \in \mathbb{N} \) such that \( k > |p| \). Then, for \( n > k \) we have
     \[ 0 \leq \frac{p^n}{n!} = \frac{|p|^n}{|n|!} = k^n = \frac{n!}{n!} \leq k^n \]. Expand this and split it in two halves. In the first half, keep the terms where the denominator is less than \( k \) and in the second half the remaining terms. Then, use the squeeze theorem.

4. Consider \((x_n)\) a sequence of real numbers such that \( \lim_{n \to \infty} x_n = x \) where \( x > 0 \), prove there exists an integer \( N > 0 \) such that \( n \geq N \implies x_n > 0 \).
   Hint: since \( x_n \to x \), one can find \( N > 0 \) such \( n \geq N \implies |x_n - x| < \frac{x}{2} = \frac{x}{2} \).

5. Consider \((x_n)\) a sequence of real numbers such that \( x_n \geq 0 \) for any \( n \). If \( x \) is a partial limit of \((x_n)\), prove that \( x \geq 0 \). Prove the same result if \( x = \lim x_n \). Use this to show that if \( x_n \leq y_n \) then \( \lim x_n \leq \lim y_n \).
   Hint: for the partial limit question, use the definition of partial limit. For the limit question, do a proof by contradiction. For the last question, apply the limit question to the sequence \( y_n - x_n \).

6. Consider \((x_n)\) a sequence of real numbers and let \( x \in \mathbb{R} \). Prove the two conditions below are equivalent.
   Hint: Don’t forget to do both directions. Just use the definition.
   
   - (a) \( x_n \to x \) as \( n \to \infty \).
   - (b) \( |x_n - x| \to 0 \) as \( n \to \infty \).

7. Consider \((x_n)\) a sequence of real numbers and let \( x \in \mathbb{R} \). If \( \lim x_n = x \), prove that \( \lim |x_n| = |x| \).
   Hint: use this version of the triangle inequality: \( ||x_n| - |x|| \leq |x_n - x| \).
8. Show by examples that if \(|x_n| \to |x|\) then \(x_n\) does not necessarily converge. If it does converge, it does not necessarily converge to \(x\).
   Hint: think of sequences we’ve used in the early examples.

9. Consider \((x_n)\) a sequence of real numbers and let \(x \in \mathbb{R}\). Suppose that 
   \(\lim x_n = x\). Define \(y_n = x_{n+p}\) for some integer \(p\). Prove that \(\lim y_n = x\).
   Hint: use the definition.

10. Given that \(a_n \leq b_n\) for every \(n\) and that \(\lim a_n = \infty\), prove that \(\lim b_n = \infty\).
    Hint: use the definition.

11. Suppose that \((a_n)\) and \((b_n)\) are sequences of real numbers such that 
    \(|a_n - b_n| < 1\) for every \(n\). Prove that if \(\infty\) is a partial limit of \((a_n)\) 
    then \(\infty\) is also a partial limit of \((b_n)\).
    Hint: use the definition.

12. Two sequences are said to be eventually close if \(\forall \epsilon > 0, \exists N > 0 : n \geq N \implies |a_n - b_n| < \epsilon\).

   (a) Prove that if two sequences \((a_n)\) and \((b_n)\) are eventually close and if 
   a number \(x\) is the limit of the sequence \((a_n)\) then \(x\) is also the limit 
   of the sequence \((b_n)\).
   Hint: use the definition and using properties of absolute value, rewrite 
   \(|b_n - x|\) as a sum of absolute values also involving \(a_n\).

   (b) Prove that if two sequences \((a_n)\) and \((b_n)\) are eventually close and if 
   a number \(x\) is a partial limit of the sequence \((a_n)\) then \(x\) is also a 
   partial limit of the sequence \((b_n)\).
   Hint: same hint as part a.

13. Determine if the sequences below have limits (finite or infinite). In each 
    case, if a limit (finite or infinite) exists, prove it using the definition of the 
    limit of a sequence.

   (a) \(\frac{n-1}{n+1}\)
   Hint: straight application of the definition.
   Answer: 1.

   (b) \(\frac{3n^2+1}{n^2+1}\)
   Hint: straight application of the definition, then simplify the expression 
   you get using techniques explained in the previous section.
   Answer: 3.

   (c) \(\frac{5n^3+n-4}{2n^3+3}\)
   Hint: straight application of the definition, then simplify the expression 
   you get using techniques explained in the previous section.
   Answer: \(\frac{5}{2}\).
4.3. LIMIT OF A SEQUENCE: THEOREMS

(d) \( \frac{n^3}{n + 1} \)

Hint: straight application of the definition, then simplify the expression you get using techniques explained in the previous section.

Answer: \( \infty \).

(e) \( \frac{4n^2 + 1}{n^3 + n} \)

Hint: straight application of the definition, then simplify the expression you get using techniques explained in the previous section.

Answer: 0.

(f) \( \frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{n}{n^2} \)

Hint: first combine all the fractions to simplify them, then it is a straight application of the definition.

Answer: \( \frac{1}{2} \).

(g) \( (-1)^{2n-1} \left( 1 - \frac{1}{2^n} \right) \)

Hint: use Bernoulli’s inequality.

Answer: -1.

(h) \( \left( 1 - \frac{1}{2} \right) \left( 1 - \frac{1}{3} \right) \left( 1 - \frac{1}{4} \right) \ldots \left( 1 - \frac{1}{n} \right) \)

Hint: Evaluate this product for several values of \( n \), then make a conjecture of what a simpler formula for this product should be. Prove your conjecture. Then, use the simpler formula to find the limit using the definition of a limit.

Answer: 0.

(i) \( \frac{5^n}{2n} \)

Hint: use techniques of this section shown in the examples.

Answer: \( \infty \).

14. Prove that if \( a > 1 \) then \( a^2 > a > \sqrt{a} > 1 \)

Hint: just do it!

15. Prove that \( \frac{n}{2^n} \to 0 \), \( \frac{n^2}{2^n} \to 0 \), \( \frac{n^3}{2^n} \to 0 \), \( \frac{n^4}{2^n} \to 0 \)

Hint: use the binomial theorem, see examples in this section.

16. Prove theorem 4.3.1

Hint: we’ve already proven this. See chapter on real numbers.

17. Finish proving theorem 4.3.8

Hint: look at the proofs in the notes, use similar techniques.

18. Let \( (x_n) \) and \( (y_n) \) be two sequences. Assume further that \( \lim x_n = x \) and \( y \) is a partial limit of \( (y_n) \). Prove that \( x + y \) is a partial limit for \( (x_n + y_n) \).

Hint: just use the various definitions involved.
19. State and prove similar results for subtraction, multiplication and division.  
Hint: just use the various definitions involved.

20. Give an example of two divergent sequences \((x_n)\) and \((y_n)\) such that \((x_n + y_n)\) is convergent.  
Hint: none.

21. Give an example of two sequences \((x_n)\) and \((y_n)\) such that \(x_n \rightarrow 0, y_n \rightarrow \infty\) and:
   (a) \(x_n y_n \rightarrow 0\)
       Hint: none.
   (b) \(x_n y_n \rightarrow c\) where \(0 < c < \infty\)
       Hint: none.
   (c) \(x_n y_n \rightarrow \infty\)
       Hint: none.
   (d) \((x_n y_n)\) is bounded but has no limit.
       Hint: none.

22. Let \((x_n)\) be a sequence of real numbers such that \(x_n \rightarrow 0\).  
Prove that
\[
\frac{x_1 + x_2 + \ldots + x_n}{n} \rightarrow 0
\]
Hint: since \(x_n \rightarrow 0\), one can find \(N_1\) such that \(n \geq N_1 \implies |x_n| < \frac{\varepsilon}{2}\).
Write \(x_1 + x_2 + \ldots + x_n\) as \((x_1 + x_2 + \ldots + x_{N_1}) + (x_{N_1+1} + \ldots + x_n)\).

23. Let \((x_n)\) be a sequence of real numbers such that \(x_n \rightarrow x\) for some real number \(x\).  
Prove that
\[
\frac{x_1 + x_2 + \ldots + x_n}{n} \rightarrow x
\]
Hint: Apply the result of the previous problem to \((y_n)\) where \(y_n = x_n - x\).

24. Prove that \(|x_n| \rightarrow 0 \iff x_n \rightarrow 0\).
   Hint: don’t forget to do both directions. For one direction, use \(-|x_n| \leq x_n \leq |x_n|\), for the other, use the properties of limits.

25. Show by examples that if \((x_n)\) and \((y_n)\) are divergent sequences then \((x_n + y_n)\) is not necessarily divergent. Do the same for \((x_n y_n)\)
   Hint: none.

26. Show that if \((x_n) \rightarrow 0\) and \((y_n)\) is bounded then \((x_n y_n) \rightarrow 0\).
   Hint: use the definitions and the following; since \(y_n\) is bounded, there exists \(M > 0\) such that \(|y_n| < M\). Choose \(N\) so large that \(n \geq N \implies |x_n| < \frac{\varepsilon}{M+1}\). Then, if \(n \geq N\), we have ...
27. Prove that if \((x_n)\) is a sequence in a set \(S \subseteq \mathbb{R}\), then every finite partial limit of \((x_n)\) must belong to \(\overline{S}\).
Hint: this is not difficult. Just write down the various definitions involved and what you have to prove and you’ll almost be done.

28. Prove that if \(S \subseteq \mathbb{R}\) and \(x \in \overline{S}\) then there exists a sequence \((x_n)\) in \(S\) such that \(x_n \to x\).
Hint: build the sequence by using the definition of \(\overline{S}\) letting \(\delta = \frac{1}{n}\).

29. Let \(S \subseteq \mathbb{R}\). Prove that the following conditions are equivalent:
Hint: don’t forget to do both directions. One direction is just the definition. For the other direction, prove that no real number can be an upper bound of \(S\) using the fact that \(x_n \to \infty\).

(a) \(S\) is unbounded above.
(b) There exists a sequence \((x_n)\) in \(S\) such that \(x_n \to \infty\).