The Binomial Series

Dr. Philippe B. Laval
Kennesaw State University
November 19, 2012

Abstract

This hand reviews the binomial theorem and presents the binomial series.

1 The Binomial Series

1.1 The Binomial Theorem

This theorem deals with expanding expressions of the form \((a + b)^k\) where \(k\) is a positive integer. In the case \(k = 2\), the result is a known identity

\[(a + b)^2 = a^2 + 2ab + b^2\]

It is also easy to derive an identity for \(k = 3\).

\[(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\]

There is also a formula for \(k\) in general. That formula is known as the Binomial Theorem. Before we state it, let us explain it a little bit. \((a + b)^k\) will be a sum of terms. Each term will contain a coefficient as well as powers of \(a\) and \(b\). More precisely, we will have

\[(a + b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{3!}a^{k-3}b^3 + \ldots + \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}a^{k-n}b^n + \ldots + kab^{k-1} + b^k\]

We see from the formula that the powers of \(a\) and \(b\) are of the form \(a^ib^j\) where \(i\) decreases from \(k\) to 0 and \(j\) increases from 0 to \(k\). The coefficients \(\frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}\) which appear in this expansion are called binomial coefficients. We use a special notation for them.
Definition 1 (Binomial coefficients) The binomial coefficients, denoted $\binom{k}{n}$, are defined by:

$$\binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!} \text{ if } n \geq 1$$

$$\binom{k}{0} = 1$$

Remark 2 The numerator of the fraction in the definition has exactly $n$ terms. This is helpful when figuring out the coefficients.

This notation allows us to write:

Theorem 3 (Binomial Theorem) Suppose that $k$ is a positive integer. Then

$$(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n$$

Example 4 Find $\binom{4}{n}$ for $n = 0, 1, 2, 3, 4$.

- $\binom{4}{0} = 1$
- $\binom{4}{1} = 4$
- $\binom{4}{2} = \frac{4 \times 3}{2!} = 6$
- $\binom{4}{3} = \frac{4 \times 3 \times 2}{3!} = 4$
- $\binom{4}{4} = \frac{4 \times 3 \times 2 \times 1}{4!} = 1$

Example 5 Expand $(a+b)^4$.

From the binomial theorem, we have

$$(a+b)^4 = \sum_{n=0}^{4} \binom{4}{n} a^{4-n} b^n$$

$$= \binom{4}{0} a^4 + \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 + \binom{4}{3} ab^3 + \binom{4}{4} b^4$$

$$= a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$$

For what follows, we will be interested in expanding $(1+x)^k$. In the case $k$ is a positive integer, the binomial theorem gives us

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$$
1.2 The Binomial Series

The binomial series extends the binomial theorem for cases when \( k \) is not an integer. For example, how would we expand \((1 + x)^{1/2}\)? In other words, given \( f(x) = (1 + x)^k \), for any \( k \), what is a Maclaurin series for \( f \)? We derive it like any other Maclaurin series. Remembering that a Maclaurin series for \( f \) is

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,
\]

we have

\[
(1 + x)^k = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.
\]

So, we need to find \( f^{(n)}(0) \).

\[
\begin{align*}
f(x) &= (1 + x)^k, & f(0) &= 1 \\
f'(x) &= k (1 + x)^{k-1}, & f'(0) &= k \\
f''(x) &= k (k - 1) (1 + x)^{k-2}, & f''(0) &= k (k - 1) \\
f'''(x) &= k (k - 1) (k - 2) (1 + x)^{k-3}, & f'''(0) &= k (k - 1) (k - 2) \\
& \vdots \\
f^{(n)}(x) &= k (k - 1) \cdots (k - n + 1) (1 + x)^{k-n}, & f^{(n)}(0) &= k (k - 1) \cdots (k - n + 1)
\end{align*}
\]

Therefore, the Maclaurin series for \((1 + x)^k\) is

\[
(1 + x)^k = \sum_{n=0}^{\infty} \frac{k (k - 1) \cdots (k - n + 1)}{n!} x^n.
\]

This series is known as the binomial series. To study its convergence, we use the ration test.

\[
\frac{|a_{n+1}|}{|a_n|} = \frac{|k (k - 1) \cdots (k - n)| |x|^{n+1}}{(n + 1)!} \frac{n!}{|k (k - 1) \cdots (k - n + 1)| |x|^n} = \frac{|k - n|}{n + 1} |x|
\]

So, since

\[
\lim_{n \to \infty} \frac{|k - n|}{n + 1} = 1
\]

we have

\[
\lim_{n \to \infty} \frac{|k - n|}{n + 1} |x| = |x|
\]

By the ratio test, this series converges if \(|x| < 1\). Convergence at the endpoints depends on the values of \( k \) and needs to be checked every time.
Definition 6 (Binomial Series) If \( |x| < 1 \) and \( k \) is any real number, then

\[
(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n
\]

where the coefficients \( \binom{k}{n} \) are the binomial coefficients.

Remark 7 This formula is very similar to the formula in the binomial theorem. The only difference is that in the binomial theorem, we have a finite sum. In this case, we have an infinite sum.

Remark 8 In the case \( k \) is a positive integer, this formula is the same as the formula of the binomial theorem. In this case, \( \binom{k}{n} = 0 \) whenever \( k > n \) (why?).

Example 9 Expand \( \frac{1}{1+x} \) as a power series.

We have already done this using substitution and the power series of \( \frac{1}{1-x} \). We found

\[
\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n
\]

We just derive the series to illustrate our new technique. From the formula above, we will have

\[
\frac{1}{1+x} = (1 + x)^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} x^n
\]

We compute

\[
\binom{-1}{n} = (-1)(-1-1)(-1-2)\ldots(-1-n+1)
\]

Remember that the numerator has \( n \) factors. So, we get

\[
\binom{-1}{n} = \frac{(-1)^n (1)(2)(3)\ldots(n)}{n!} = (-1)^n
\]

Therefore,

\[
\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n
\]

Which is the same formula as we had found using a different method.
Example 10 Expand $\frac{1}{\sqrt{1 + x}}$ as a power series.

First, we notice that

$$\frac{1}{\sqrt{1 + x}} = (1 + x)^{-\frac{1}{2}}$$

So, we use the formula above in the case $k = -\frac{1}{2}$. We obtain:

$$\frac{1}{\sqrt{1 + x}} = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right) \binom{n}{2^n} x^n$$

where

$$\binom{-\frac{1}{2}}{0} = 1$$

and if $n \geq 1$,

$$\binom{-\frac{1}{2}}{n} = \frac{-\frac{1}{2} \left( -\frac{1}{2} - 1 \left( -\frac{1}{2} - 2 \cdots \left( -\frac{1}{2} - n + 1 \right) \right)\right)}{n!}$$

$$= \frac{\left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \left( -\frac{5}{2} \right) \cdots \left( -\frac{2n-1}{2} \right)}{n!}$$

The numerator has $n$ factors. We get

$$\binom{-\frac{1}{2}}{n} = (-1)^n \frac{(1)(3)(5)\cdots(2n-1)}{2^n n!}$$

Therefore,

$$\frac{1}{\sqrt{1 + x}} = 1 - \frac{1}{2} x + \frac{(1)(3)}{2^2 2!} x^2 - \frac{(1)(3)(5)}{2^3 3!} x^3 \cdots$$

1.3 Problems

Do # 1, 3, 7, 9 on page 625