6.9 Applications of Infinite Series

The purpose of this section is to show the reader how Taylor series can be used to approximate functions. The approximation can then be used to either evaluate a function at specific values of x, to integrate or to differentiate the function. Of course, we would only use this technique with functions for which the traditional calculus methods do not work. For example, we may need to compute $\int_0^1 e^{-x^2}dx$. This cannot be done using the integration techniques learned in a traditional calculus class since e^{-x^2} does not have an antiderivative which can be expressed in terms of elementary functions. Another application might be to approximate $\sin(0.01)$, without a calculator.

The difficulty does not lie in the series representation of a given function, we now know how to represent functions as power series. However, series have infinitely many terms, for practical purposes, we can only use a finite number of them. Thus, we replace the infinite series by the corresponding Taylor polynomial (see definition 6.8.3) of order n (T_n), for some n. We then use the Taylor polynomial instead of the function. This can be used to evaluate a function, integrate a function or differentiate a function. However, when we perform the following approximation

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

there are several questions to answer before we can carry it out:

1. How do we pick a, the number around which the series is centered?

2. How do we pick n so the Taylor polynomial T_n approximates the given function f within the desired accuracy?

Answer to question 1 There are several factors to take under consideration. First, since we have to evaluate $f^{(n)}(a)$, a must be picked so we can do this evaluation easily. Second, you will recall that the accuracy of the approximation decreases as x gets further away from a. Therefore, we need to pick a so that the values at which $f(x)$ will be approximated are in the domain of the series, and not too far from a. For example, if we had to approximate $\sin(.001)$, then $a = 0$ would be a good choice because it satisfies both criteria.

Answer to question 2 Once we have selected a and have a series representation for f, we use the techniques studied to approximate a series within the required error.

The examples below illustrate these applications.

Example 6.9.1 *Find the nth order Taylor polynomial for $f(x) = \cos x$ when $n = 2, 3, 4, 5, 6$. Sketch the graph of $\cos x$ as well as the Taylor polynomials*

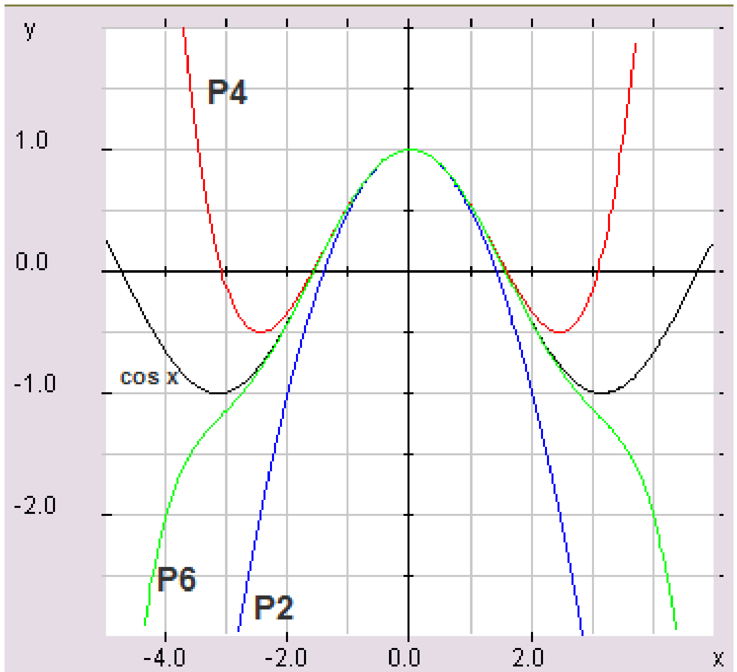

Figure 6.6: $\cos x$, $P_2(x)$, $P_4(x)$ and $P_6(x)$

found.
We already know the power series for $\cos x$*.*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}....$$

So,

$$\begin{aligned} P_2(x) &= 1 - \frac{x^2}{2!} \\ P_3(x) &= 1 - \frac{x^2}{2!} \\ P_4(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\ P_5(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\ P_6(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \end{aligned}$$

You will note that because every other coefficient in the series expansion of $\cos x$ *is* 0, $P_3 = P_2$, $P_5 = P_4$*. The graph of these polynomials is shown on figure 6.6.*

Remark 6.9.2 *You will notice that as n increases, P_n gets closer to the graph of $\cos x$. In other words, the accuracy of our approximation increases with n. However, P_n is a good approximation for $\cos x$ in a neighborhood of 0, as we move away from 0, P_n gets further and further away from $\cos x$. This is important. When one approximates a function with a Taylor polynomial about a, the approximation is good only for values of x close to a.*

Example 6.9.3 *Approximate $\cos 0.01$ with an error less than 10^{-20}.*
First, we note that since .01 is close to 0, we can use a Taylor polynomial centered at 0 to approximate $\cos(.01)$. Therefore, using the Taylor polynomial for $\cos x$ centered at 0, and replacing x by 0.01, we get:

$$\cos 0.01 = \sum_{i=0}^{n} (-1)^i \frac{0.01^{2i}}{(2i)!}$$

We need to find n so that if we approximate $\sum_{i=0}^{\infty} (-1)^i \frac{0.01^{2i}}{(2i)!}$ by $\sum_{i=0}^{n} (-1)^i \frac{0.01^{2i}}{(2i)!}$, the error is less than 10^{-20}. We notice that $\sum_{i=0}^{\infty} (-1)^i \frac{0.01^{2i}}{(2i)!}$ is an alternating series with $b_n = \frac{0.01^{2n}}{(2n)!}$, so we know how to estimate its error. The error is always less than b_{n+1}. So, if we want the error to be less than 10^{-20}, it is enough to solve:

$$\begin{aligned} b_{n+1} &< 10^{-20} \\ \frac{0.01^{2(n+1)}}{(2n+2)!} &< 10^{-20} \end{aligned}$$

We solve this by trying various values of n . The table below shows this procedure:

n	$\frac{0.01^{2(n+1)}}{(2n+2)!}$
1	4×10^{-10}
2	1×10^{-15}
3	2×10^{-21}

We see that $n = 3$ is enough. Therefore:

$$\begin{aligned} \cos 0.01 &\approx \sum_{n=0}^{3} (-1)^n \frac{0.01^{2n}}{(2n)!} \\ &\approx 0.999950000416665 \end{aligned}$$

Example 6.9.4 *Approximate* $\int_0^1 e^{-x^2}dx$ *with an error less than* 0.001. *First, we find a series representation for* e^{-x^2}. *Since*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

it follows that

$$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

Therefore

$$\int e^{-x^2}dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\,n!}$$

and therefore

$$\begin{aligned} \int_0^1 e^{-x^2}dx &= \left. \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\,n!} \right|_0^1 \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)\,n!} \end{aligned}$$

This is an alternating series $\sum_{n=0}^{\infty} (-1)^n b_n$ *with* $b_n = \frac{1}{(2n+1)\,n!}$. *From our knowledge of alternating series, we know that if we approximate* $\sum_{i=0}^{\infty} (-1)^i \frac{1}{(2i+1)\,i!}$ *by* $\sum_{i=0}^{n} (-1)^i \frac{1}{(2i+1)\,i!}$, *the error will be less than* $b_{n+1} = \frac{1}{(2n+3)\,(n+1)!}$. *So, we find* n *such that*

$$\frac{1}{(2n+3)\,(n+1)!} < .001$$

We try several values of n, *we find that when* $n = 3$, $\frac{1}{(2n+3)\,(n+1)!} = .00463$ *and when* $n = 4$, $\frac{1}{(2n+3)\,(n+1)!} = .0007576$. *So,*

$$\sum_{n=0}^{4} (-1)^n \frac{1}{(2n+1)\,n!} = 0.747\,49$$

gives us the desired approximation.

6.9.1 Problems

1. Approximate $\sin 1$ (1 is in radians) with an error less than .01. (hint: the series for $\sin x$ is an alternating series. Use the result about the error when approximating alternating series).

2. For small values of x (in radians), it is customary to use the approximation $\sin x \approx x$. Find the range of values of x for which this approximation has an error less than .1. (hint: the series for $\sin x$ is an alternating series. Use the result about the error when approximating alternating series).

3. The purpose of this problem is to evaluate $\int_0^1 \cos\left(x^2\right) dx$. Since $\cos\left(x^2\right)$ does not have an antiderivative which can be expressed in terms of elementary functions, we approximate this integral using infinite series. We do it in several steps.

   (a) Find the series representation for $\cos\left(x^2\right)$, find its interval of convergence.

   (b) Find a series representation for $\int \cos\left(x^2\right) dx$, find its interval of convergence.

   (c) Evaluate $\int_0^1 \cos\left(x^2\right) dx$ with an error less than .001.

4. The force due to gravity on an object with mass m at a height h above the surface of the earth is

$$F = \frac{mgR^2}{(R+h)^2}$$

   where R is the radius of the earth and g is the acceleration due to gravity.

   (a) Show that $F = \dfrac{mg}{\left(1+\dfrac{h}{R}\right)^2}$.

   (b) Using the formula in part a), express F as a series in powers of $\dfrac{h}{R}$. (hint: replace $\dfrac{h}{R}$ by x, express F as a series in powers of x, then replace x back with $\dfrac{h}{R}$).

   (c) You may have learned in physics that the force of gravity on an object which is close to the surface of the earth was $F = mg$. This was an approximation of the exact formula given in this exercise. Looking at the series representation you obtained in part b), explain why this approximation is justified when the object is close to the surface of the earth. (hint: if the object is close to the surface of the earth, what is the size of h relative to the size of R)?

   (d) Estimate the range of values of h for which this approximation is accurate with an error less that .01. You may use $R = 6400$ kilometers, $g = 9.8\ m/s^2$.

6.9.2 Answers

1. Approximate $\sin 1$ (1 is in radians) with an error less than .01. (hint: the series for $\sin x$ is an alternating series. Use the result about the error when approximating alternating series).

$$\sin(1) \approx .83$$

2. For small values of x (in radians), it is customary to use the approximation $\sin x \approx x$. Find the range of values of x for which this approximation has an error less than .1. (hint: the series for $\sin x$ is an alternating series. Use the result about the error when approximating alternating series).
The approximation $\sin x \approx x$ with an error of at most 0.1 is valid when $x \in (-.84, .84)$

3. The purpose of this problem is to evaluate $\int_0^1 \cos\left(x^2\right) dx$. Since $\cos\left(x^2\right)$ does not have an antiderivative which can be expressed in terms of elementary functions, we approximate this integral using infinite series. We do it in several steps.

   (a) Find the series representation for $\cos\left(x^2\right)$, find its interval of convergence.

$$\cos\left(x^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} \text{ valid in } (-\infty, \infty)$$

   (b) Find a series representation for $\int \cos\left(x^2\right) dx$, find its interval of convergence.

$$\int \cos\left(x^2\right) dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{4n+1}}{4n+1} + C \text{ valid in } (-\infty, \infty)$$

   (c) Evaluate $\int_0^1 \cos\left(x^2\right) dx$ with an error less than .001.

$$\begin{aligned} \int_0^1 \cos\left(x^2\right) dx &\approx \sum_{n=0}^{2} \frac{(-1)^n}{(2n)!} \frac{1}{4n+1} \\ &\approx 0.905 \end{aligned}$$

4. The force due to gravity on an object with mass m at a height h above the surface of the earth is

$$F = \frac{mgR^2}{(R+h)^2}$$

where R is the radius of the earth and g is the acceleration due to gravity.

(a) Show that $F = \dfrac{mg}{\left(1+\dfrac{h}{R}\right)^2}$.

Factor R^2 in from the denominator.

(b) Using the formula in part a), express F as a series in powers of $\dfrac{h}{R}$. (hint: replace $\dfrac{h}{R}$ by x, express F as a series in powers of x, then replace x back with $\dfrac{h}{R}$).

$$\begin{aligned} F &= mg\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{h}{R}\right)^n \\ &= mg\left(1-2\frac{h}{R}+3\left(\frac{h}{R}\right)^2-4\left(\frac{h}{R}\right)^3 ...\right) \end{aligned}$$

(c) You may have learned in physics that the force of gravity on an object which is close to the surface of the earth was $F = mg$. This was an approximation of the exact formula given in this exercise. Looking at the series representation you obtained in part b), explain why this approximation is justified when the object is close to the surface of the earth. (hint: if the object is close to the surface of the earth, what is the size of h relative to the size of R)?

Use the hint.

(d) Estimate the range of values of h for which this approximation is accurate with an error less that .01. You may use $R = 6400$ kilometers, $g = 9.8\ m/s^2$.

The approximation $F \approx mg$ is valid with an error less than 0.01 as long as the object is less than 32 km from the surface of the earth.