Telescoping Sums

Dr. Philippe B. Laval
Kennesaw State University

Abstract

This hand out is a description of the technique known as telescoping sums, which is used when studying the convergence of some series.

1 Telescoping sums

1.1 Introduction

The technique we are about to describe applies to series of the form $\sum_{i=1}^{\infty} \frac{1}{i(i+p)}$ where p is some positive integer. It is used when we are studying the convergence of these series. For example, we would use it to study the series $\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$, $\sum_{i=1}^{\infty} \frac{1}{i(i+2)}$, $\sum_{i=1}^{\infty} \frac{1}{i(i+3)}$, ... It involves two steps. First, we write the general term of the series as a difference of two fractions, using partial fraction decomposition. Second, we find and simplify the sequence of partial sums, as most of its terms will cancel.

2 Partial Fraction Decomposition.

Theorem 1 $\frac{1}{i(i+p)} = \frac{1}{ip} - \frac{1}{p(i+p)} = \frac{1}{p}\left(\frac{1}{i} - \frac{1}{i+p}\right)$

Proof. *This is easily done using the techniques of partial fraction decomposition. First, we notice that*

$$\frac{1}{i(i+p)} = \frac{A}{i} + \frac{B}{i+p} \tag{1}$$

If we multiply each side by the denominator of the fraction on the left, we obtain

$$1 = A(i+p) + Bi$$

When $i = 0$, we obtain

$$\begin{aligned} Ap &= 1 \\ A &= \frac{1}{p} \end{aligned}$$

When $i = -p$, we obtain

$$\begin{aligned} -Bp &= 1 \\ B &= -\frac{1}{p} \end{aligned}$$

Replacing those values in equation 1 gives the desired result. ■

Example 2 *If $p = 1$, we have*

$$\frac{1}{i\,(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

Example 3 *If $p = 2$, we have*

$$\frac{1}{i\,(i+2)} = \frac{1}{2}\left(\frac{1}{i} - \frac{1}{i+2}\right)$$

Example 4 *If $p = 3$, we have*

$$\frac{1}{i\,(i+3)} = \frac{1}{3}\left(\frac{1}{i} - \frac{1}{i+3}\right)$$

3 Simplifying the Sequence of Partial Sums

Instead of studying $\sum_{i=1}^{\infty} \frac{1}{i\,(i+p)}$, we study $\sum_{i=1}^{\infty} \frac{1}{p}\left(\frac{1}{i} - \frac{1}{i+p}\right)$. In fact, because we will prove that $\sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+p}\right)$ converges, it is enough to study $\sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+p}\right)$ since

$$\sum_{i=1}^{\infty} \frac{1}{p}\left(\frac{1}{i} - \frac{1}{i+p}\right) = \frac{1}{p}\sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+p}\right) \tag{2}$$

We now concentrate on $\sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+p}\right)$. To study it, we use its sequence of partial sums, (S_n). By definition,

$$S_n = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+p}\right)$$

We first look at the special cases $p = 1$ and $p = 2$. We then generalize our result for any positive integer p.

3.1 Case $p = 1$

In this case,

$$S_n = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right)$$

The key here is to notice that $\frac{1}{i}$ and $\frac{-1}{i+1}$ will generate the same values with opposite signs. Hence, they will cancel. Of course, these same values will be generated for different values of i. More precisely, they will generate the same values for values of i which are 1 unit apart. For a certain value of i, $\frac{1}{i}$ generates the same value that $\frac{1}{i+1}$ generated for the previous value of i. For example, when $i = 4$, $\frac{1}{i} = \frac{1}{4}$ which is the same value that $\frac{1}{i+1}$ gives when $i = 3$. This means that everything will cancel except the value generated by $\frac{1}{i}$ for the starting value of i, and the last value of $\frac{1}{i+1}$ for the ending value of i. When $i = 1$, $\frac{1}{i} = 1$. $\frac{1}{i+1} = 1$ when $i = 0$, but i is never 0. Similarly, when $i = n$, $\frac{1}{i+1} = \frac{1}{n+1}$ which can only be generated by $\frac{1}{i}$ for $i = n+1$. But i is never $n+1$. Therefore, everything cancels except $\frac{1}{i}$ for the starting value of i and $\frac{-1}{i+1}$ for the ending value of i. Therefore, we see that

$$S_n = 1 - \frac{1}{n+1}$$

This can be verified if we expand S_n

$$\begin{aligned} S_n &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

Since

$$\lim_{n \to \infty} S_n = 1$$

It follows that

$$\sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1} \right) = 1$$

3.2 Case $p = 2$

In this case,

$$S_n = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+2} \right)$$

This is similar to above, but this time, $\frac{1}{i}$ and $\frac{-1}{i+2}$ will generate the same values with opposite signs for values of i which are 2 units apart. Therefore, using the same reasoning as above, we see that the first two values generated by $\frac{1}{i}$ and the last two values generated by $\frac{-1}{i+2}$ will not cancel. Everything else will cancel. Hence,

$$S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

We see that

$$\lim_{n\to\infty} S_n = 1 + \frac{1}{2}$$

Therefore

$$\sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+2}\right) = 1 + \frac{1}{2}$$

3.3 General case

In this case,

$$S_n = \sum_{i=1}^{n}\left(\frac{1}{i} - \frac{1}{i+p}\right)$$

This is similar to above, but this time, $\frac{1}{i}$ and $\frac{-1}{i+p}$ will generate the same values with opposite signs for values of i which are p units apart. Therefore, using the same reasoning as above, we see that the first p values generated by $\frac{1}{i}$ and the last p values generated by $\frac{-1}{i+2}$ will not cancel. Everything else will cancel. Hence,

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{p} - \frac{1}{n+1} - \frac{1}{n+2} - ... - \frac{1}{n+p}$$

We see that

$$\lim_{n\to\infty} S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{p}$$

Therefore

$$\begin{aligned}\sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+p}\right) &= 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{p} \\ &= \sum_{i=1}^{p}\frac{1}{i}\end{aligned}$$

3.4 Conclusion

The above has shown us that if p is any positive integer, then

$$\sum_{i=1}^{\infty}\left(\frac{1}{i}-\frac{1}{i+p}\right)=\sum_{i=1}^{p}\frac{1}{i} \tag{3}$$

Our original problem was to study $\sum_{i=1}^{\infty}\frac{1}{i\left(i+p\right)}$. We saw above that

$$\sum_{i=1}^{\infty}\frac{1}{i\left(i+p\right)}=\sum_{i=1}^{\infty}\frac{1}{p}\left(\frac{1}{i}-\frac{1}{i+p}\right)$$

Since $\sum_{i=1}^{\infty}\left(\frac{1}{i}-\frac{1}{i+p}\right)$ converges, we have by equation 2

$$\sum_{i=1}^{\infty}\frac{1}{i\left(i+p\right)}=\frac{1}{p}\sum_{i=1}^{\infty}\left(\frac{1}{i}-\frac{1}{i+p}\right)$$

Using equation 3, we obtain the following theorem

Theorem 5 *If p is a positive integer, then*

$$\sum_{i=1}^{\infty}\frac{1}{i\left(i+p\right)}=\frac{1}{p}\sum_{i=1}^{p}\frac{1}{i} \tag{4}$$

Corollary 6 *If p is a positive integer, and C is a constant, then*

$$\sum_{i=1}^{\infty}\frac{C}{i\left(i+p\right)}=\frac{C}{p}\sum_{i=1}^{p}\frac{1}{i} \tag{5}$$

Example 7 *Study the convergence of* $\sum_{i=1}^{\infty}\frac{1}{i\left(i+1\right)}$
This is a series like the one in equation 4 with $p=1$. Therefore,

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{i\left(i+1\right)} &= \frac{1}{1}\sum_{i=1}^{1}\frac{1}{i}\\ &= 1\end{aligned}$$

You will recall that we had already derived this result as an example in the handout on series.

Example 8 *Study the convergence of* $\sum_{i=1}^{\infty} \frac{1}{i\,(i+2)}$
This is a series like the one in equation 4 with $p = 2$. *Therefore,*

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{1}{i\,(i+2)} &= \frac{1}{2}\sum_{i=1}^{2} \frac{1}{i} \\ &= \frac{1}{2}\left(1 + \frac{1}{2}\right) \\ &= \frac{3}{4}\end{aligned}$$

You will recall that we had already found this result as part as one of the exercises assigned.

Example 9 *Study the convergence of* $\sum_{i=1}^{\infty} \frac{5}{i\,(i+3)}$
This is a series like the one in equation 5 with $p = 3$ *and* $C = 5$. *Therefore,*

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{5}{i\,(i+3)} &= \frac{5}{3}\sum_{i=1}^{3} \frac{1}{i} \\ &= \frac{5}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) \\ &= \frac{55}{18}\end{aligned}$$