

1.7 Cylindrical and Spherical Coordinates

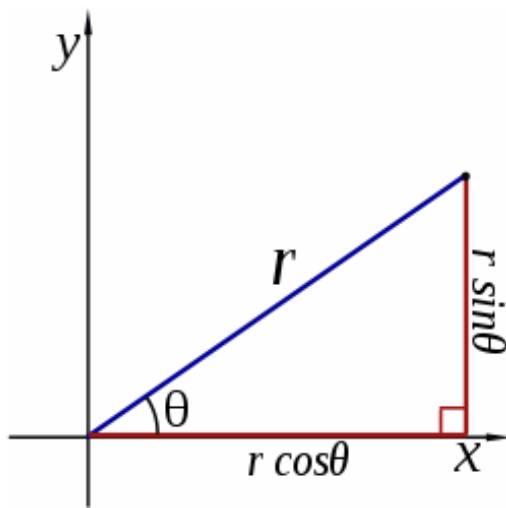
1.7.1 Review: Polar Coordinates

The polar coordinate system is a two-dimensional coordinate system in which the position of each point on the plane is determined by an angle and a distance. The distance is usually denoted r and the angle is usually denoted θ . Thus, in this coordinate system, the position of a point will be given by the ordered pair (r, θ) . These are called the **polar coordinates**

These two quantities r and θ , are determined as follows. First, we need some reference points. You may recall that in the Cartesian coordinate system, everything was measured with respect to the coordinate axes. In the polar coordinate system, everything is measured with respect a fixed point called **the pole** and an axis called the **polar axis**. This is the equivalent of the origin in the Cartesian coordinate system. The polar axis corresponds to the positive x-axis. Given a point P in the plane, we draw a line from the pole to P . The distance from the pole to P is r , the angle, measured counterclockwise, by which the polar axis has to be rotated in order to go through P is θ . The polar coordinates of P are then (r, θ) . Figure 1.7.1 shows two points and their representation in the polar coordinate system.

Remark 79 *Let us make several remarks.*

1. Recall that a positive value of θ means that we are moving counterclockwise. But θ can also be negative. A negative value of θ means that the polar axis is rotated clockwise to intersect with P . Thus, the same point can have several polar coordinates. For example, $(2, 90)$ and $(2, -270)$ represent the same point.
2. Recall that given an angle θ , the angles $\theta + 2\pi$, $\theta + 4\pi$, ..., $\theta + 2k\pi$ where $k = 0, 1, 2, 3, \dots$ represent the same angle. Thus, (r, θ) and $(r, \theta + 2k\pi)$ where $k = 0, 1, 2, 3, \dots$ represent the same point.



3. Recall that a positive value of r means that the point is away from the pole in the direction of the positive x -axis (taking into account the rotation by θ). But r can also have a negative value. A negative value of r means that the point is away from the pole in the direction of the negative x -axis. Thus (r, θ) and $(-r, \theta + \pi)$ represent the same point.
4. The pole has infinitely many polar coordinates. They are of the form $(0, \theta)$ where θ can be anything.

Why do we need another coordinate system? Cartesian coordinates are not best suited for every shape. Certain shape, including circular ones, cannot even be represented as a function in Cartesian coordinates. These shapes are more easily represented in polar coordinates. We will see some examples below. First, we need to see how to go back and forth between Cartesian and polar coordinates. Figure 1.7.1 helps us understand how the two coordinate systems are related. We list the formulas which allow the switch between the two coordinate systems as a proposition.

Proposition 80 *The following formulas can be used to switch between Cartesian and polar coordinates.*

1. **From polar to Cartesian coordinates:** Given r and θ , we can find x and y by:

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \tag{1.16}$$

2. **From Cartesian to polar coordinates:** Given x and y , we can find r

and θ by:

$$\begin{aligned}x^2 + y^2 &= r^2 \\ \tan \theta &= \frac{y}{x}\end{aligned}\tag{1.17}$$

We now look at some examples of converting points between polar and Cartesian coordinates.

Example 81 Convert $\left(-4, \frac{2\pi}{3}\right)$ in Cartesian coordinates.

Using formulas 1.16 we have:

$$\begin{aligned}x &= -4 \cos \frac{2\pi}{3} = -4 \left(-\frac{1}{2}\right) = 2 \\ y &= -4 \sin \frac{2\pi}{3} = -4 \left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}\end{aligned}$$

So, in Cartesian coordinates, the point is $(2, -2\sqrt{3})$.

Example 82 Convert $(1, 0)$ in polar coordinates.

We don't need a formula here. This point is on the x -axis, one unit from the origin. Its polar coordinates are $(1, 0)$

Example 83 Convert $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ in polar coordinates.

Using formulas 1.17, we have:

$$r^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1$$

This gives us

$$r = \pm 1$$

and

$$\begin{aligned}\tan \theta &= \frac{x}{y} \\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= 1\end{aligned}$$

Thus

$$\theta = \frac{\pi}{4}$$

With a choice of $\theta = \frac{\pi}{4}$, we see that we must have $r = 1$. So, in polar coordinates, the point is $\left(1, \frac{\pi}{4}\right)$.

We now look at how to convert the equation of known shapes between Cartesian and polar coordinates. Recall that when we write the equation of a shape, we are writing conditions that the coordinates of a point must satisfy in order to belong to that shape. In Cartesian coordinates, this relation is between x and y . In polar coordinates, the relation will be between r and θ . Of course, the equations of the shapes you know in Cartesian coordinates will look very different in polar coordinates. To perform this, we will use the relations in equations 1.16 and 1.17.

Example 84 *What is the equation of the line $y = 2x + 5$ in polar coordinates. Rewriting in terms of r and θ , we have*

$$r \sin \theta = 2r \cos \theta + 5$$

There is not much to simplify here.

Example 85 *What is the equation of the horizontal line through $(1, 2)$ in polar coordinates?*

In Cartesian coordinates, it is $y = 2$. This gives us

$$r \sin \theta = 2$$

Example 86 *What is the equation of the circle of radius 2 centered at the origin in polar coordinates?*

In Cartesian coordinates, this equation is $x^2 + y^2 = 4$. This gives us

$$\begin{aligned} r^2 \cos^2 \theta + r^2 \sin^2 \theta &= 4 \\ r^2 (\sin^2 \theta + \cos^2 \theta) &= 4 \\ r^2 &= 4 \end{aligned}$$

or $r = 2$.

Example 87 *What is in Cartesian coordinate $\theta = 5$?*

θ is related to x and y via the tangent function. $\theta = \tan^{-1} \frac{y}{x}$. Hence, we have

$$\frac{y}{x} = \tan \theta = \tan 5$$

that is

$$y = (\tan 5)x$$

This is the equation of the line through the origin with slope $\tan 5$.

Polar coordinate can be extended to the three-dimensional space different ways. We present two here: the cylindrical coordinate system and the spherical coordinate system.

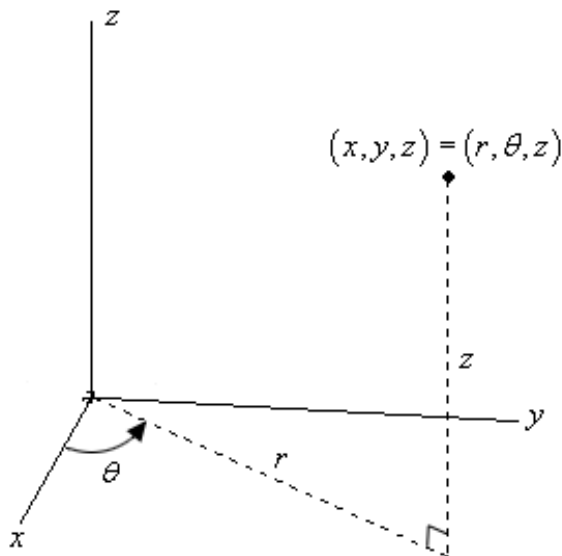


Figure 1.36: Cylindrical Coordinates

1.7.2 Cylindrical Coordinates

These are coordinates for a three-dimensional space. In this coordinate system, a point P is represented by the triple (r, θ, z) where r and θ are the polar coordinates of the projection of P onto the xy -plane and z has the same meaning as in Cartesian coordinates. Figure 1.36 shows a point in cylindrical coordinates.

Switching between Cartesian coordinates and cylindrical coordinates will use the same formulas as when switching between polar and Cartesian coordinates. We illustrate this with a few examples.

Example 88 Convert $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 5\right)$ to cylindrical coordinates.

We see that

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \tan \theta &= \frac{y}{x} \\ &= \sqrt{3} \end{aligned}$$

Thus $\theta = \frac{\pi}{3}$ and $r = 1$. Thus, the cylindrical coordinates are $(1, \frac{\pi}{3}, 5)$.

Example 89 What is the equation in cylindrical coordinates of the cone $x^2 + y^2 = z^2$.

Replacing $x^2 + y^2$ by r^2 , we obtain $r^2 = z^2$ which usually gives us $r = \pm z$. Since z can be any real number, it is enough to write $r = z$. Thus, in cylindrical coordinates, this cone is $z = r$.

Example 90 What is the equation in Cartesian coordinates of $r = 3$. What is it?

Since $r = 3$, we see that $r^2 = 9$ that is $x^2 + y^2 = 9$. This is a cylinder centered along the z -axis, of radius 3.

1.7.3 Spherical Coordinates

This is a three-dimensional coordinate system. The coordinates of a point P are given by the ordered pair (ρ, θ, φ) where:

- ρ is the distance from the origin to P . We assume $\rho \geq 0$.
- θ has the same meaning as in polar and cylindrical coordinates. There are no restrictions on θ .
- φ is the angle between the positive z -axis and the line from the origin to P . We restrict φ to $0 \leq \varphi \leq \pi$.

This is illustrated by figure 1.37.

Let us first derive the formulas to switch between Cartesian and spherical coordinates.

Using the triangle one can make with the z -axis and the line from the origin through P , we have

$$z = \rho \cos \varphi \quad (1.18)$$

and

$$r = \rho \sin \varphi \quad (1.19)$$

Also

$$z^2 + r^2 = \rho^2 \quad (1.20)$$

Using the triangle one can make with the x -axis and the line from the origin to the projection of P on the xy -plane, we have

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \quad (1.21)$$

and also

$$x^2 + y^2 = r^2 \quad (1.22)$$

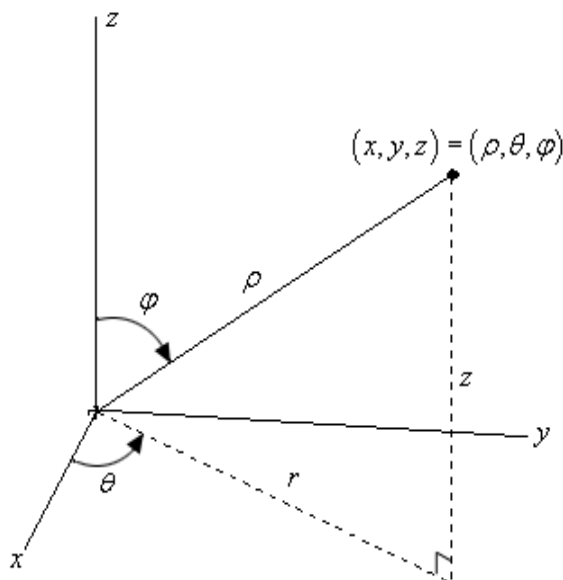


Figure 1.37: Spherical Coordinates

Combining relations 1.18, 1.19 and 1.21 gives us

$$\begin{aligned}x &= \rho \sin \varphi \cos \theta \\y &= \rho \sin \varphi \sin \theta \\z &= \rho \cos \varphi\end{aligned}$$

Combining relations 1.20 and 1.22 gives us

$$x^2 + y^2 + z^2 = \rho^2$$

In conclusion, the following relations can be used to switch between Cartesian and spherical coordinates:

$$\begin{aligned}x &= \rho \sin \varphi \cos \theta \\y &= \rho \sin \varphi \sin \theta \\z &= \rho \cos \varphi \\\rho^2 &= x^2 + y^2 + z^2\end{aligned} \tag{1.23}$$

We illustrate this with some examples.

Example 91 Convert $(-1, 1, -\sqrt{2})$ from Cartesian to spherical.

We can get ρ easily. $\rho^2 = (-1)^2 + 1^2 + (-\sqrt{2})^2 = 4$. Thus, $\rho = 2$. From

$z = \rho \cos \varphi$, we see that $\cos \varphi = \frac{-\sqrt{2}}{2}$ hence $\varphi = \cos^{-1} \frac{-\sqrt{2}}{2} = \frac{3}{4}\pi$. Finally, to find θ , we can use the equation involving x or y . Let's use y . We have $\sin \theta = \frac{y}{\rho \sin \varphi} = \frac{1}{2 \sin \frac{3\pi}{4}} = \frac{1}{2}\sqrt{2}$. Several values for θ would work. $\frac{\pi}{4}, \frac{3\pi}{4}$. However, the projection of P on the xy -plane is in the second quadrant, so we must have $\theta = \frac{3\pi}{4}$. Thus, the spherical coordinates are $(2, \frac{3\pi}{4}, \frac{3\pi}{4})$.

Example 92 Convert $(\sqrt{6}, \frac{\pi}{4}, \sqrt{2})$ from cylindrical to spherical.

Let us first note that θ is the same in both coordinates systems. Next, we can get ρ from equation 1.20. Since $r = \sqrt{6}$ and $z = \sqrt{2}$, we see that $\rho^2 = z^2 + r^2 = 8$. Thus, $\rho = 2\sqrt{2}$. Finally, we use $z = \rho \cos \varphi$ to get φ . $\cos \varphi = \frac{z}{\rho} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$. Thus, $\varphi = \frac{\pi}{3}$. Hence, the spherical coordinates are $(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3})$.

Example 93 Identify the surface given by $\rho = 2$.

Converting to Cartesian coordinates gives $x^2 + y^2 + z^2 = 4$. Hence, it is a sphere centered at the origin, with radius 2. This is important to remember. In general, the equation $\rho = c$ is the equation of the sphere of radius c centered at the origin in spherical coordinates.

Make sure you can do the above before attempting the problems.

1.7.4 Problems

Do odd # 3- 23 at the end of 9.7 in your book.