Chapter 4

Multiple Integrals

4.1 Double Integral Over Rectangular Regions

4.1.1 Quick Review of the Definite Integral

This review is intended more to introduce the notation we will use than to cover the topic of the definite integral. It is assumed the student is already familiar with it. If this is not the case, students should first review the definite integral in any calculus book, or using my notes on the web site for the class.

The definite integral was defined when we were trying to solve the area problem. Given a function \( y = f(x) \) defined for \( a \leq x \leq b \), we wanted to find the area between the graph of \( y = f(x) \), the \( x \)-axis, and the vertical lines \( x = a \) and \( x = b \). We begin by dividing the interval \([a, b]\) into \( n \) subintervals \([x_{i-1}, x_i] \), \( i = 1, \ldots, n \), of equal length. Let \( \Delta x \) denote the length of each subinterval. Clearly, \( \Delta x = \frac{b-a}{n} \). In each subinterval \([x_{i-1}, x_i] \), we pick a point we denote \( x_i^* \). We form the Riemann sum

\[
\sum_{i=1}^{n} f(x_i^*) \Delta x
\]

The definite integral was defined to be

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

In the case that \( f(x) \geq 0 \) on \([a, b]\), the integral corresponds to the area below the graph (see figure 4.1).

4.1.2 Double Integrals and Volumes

For the definite integrals we reviewed above, the function we were integrating was a function of one variable. That is a function of the form \( f : \mathbb{R} \to \mathbb{R} \). Hence, its domain is either \( \mathbb{R} \) or a subset of \( \mathbb{R} \). When we integrate, we also integrate
over portions of its domain, most of the time over closed and bounded portions of its domain, that is over intervals. The functions we are about to learn how to integrate are functions of two variables or more. Let us focus on functions on two variables for a while, that is on functions of the form $f : \mathbb{R}^2 \to R$. The domain of such functions is a subset of $\mathbb{R}^2$ that is the plane. We will be integrating over closed and bounded regions of the plane. The problem is that there are many possibilities for such regions ranging from a square or a rectangle to more complicated shapes. The region of integration is an added difficulty when dealing with multiple integrals, a difficulty we do not have for functions of one variable since we are always integrating over an interval. We will first consider integrals over a rectangular region. These are very simple. Then, we will look at more complicated regions.

In spite of this added difficulty, double integrals are defined in a manner similar to that of definite integrals. Suppose that we are given a function $f(x, y)$ defined over a closed rectangle

$$R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\} = [a, b] \times [c, d]$$

As in the case of functions of one variable, we will first assume that $f(x, y) \geq 0$ on $R$. Let $S$ be the solid which lies above $R$ and below the graph of $f$. In other words,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y), (x, y) \in R\}$$

We wish to find $V$, the volume of $S$ (see figure 4.2).

We begin by dividing $R$ into subrectangles. For this, we divide the interval $[a, b]$ into $m$ subintervals $[x_{i-1}, x_i], \ i = 1..m$, of equal length $\Delta x = \frac{b-a}{m}$. We also divide the interval $[c, d]$ into $n$ subintervals $[y_{j-1}, y_j], \ j = 1..n$, of equal length...
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Figure 4.2: Solid above $R$ below $f(x, y)$

$\Delta y = \frac{d-c}{n}$. This way, we obtain the subrectangles

\[ R_{ij} = \{(x, y) \in \mathbb{R}^2 \mid x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\} \]
\[ = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \]

The area of each $R_{ij}$ is $\Delta A = \Delta x \Delta y$ (see figure 4.3).

Next, in each subrectangle $R_{ij}$, we pick a point denoted $(x_{ij}^*, y_{ij}^*)$. We form the box with base $R_{ij}$ and height $f(x_{ij}^*, y_{ij}^*)$. Its volume is $f(x_{ij}^*, y_{ij}^*) \Delta A$ (see figure 4.4).

We can approximate the volume of $S$ by adding the volume of each box obtained (see figure 4.5). In other words, we have

\[ V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A \]

**Remark 310** The above sum is called a double Riemann sum.

As in the case of functions of one variable, our approximation should get better the larger $m$ and $n$ are. We define the double integral of $f$ over $R$ to be

**Definition 311** The **double integral** of $f(x, y)$ over the rectangle $R$ is:

\[ \iint_{R} f(x, y) \, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A \]

providing the limit exists.
Figure 4.3: Dividing a rectangle in subrectangles

Figure 4.4: Approximating the volume of a solid above a rectangle, below $f(x, y)$
Remark 312  It can be proven that the limit exists if $f$ is continuous on $R$.

Remark 313  To understand the notation, it helps to draw a parallel between the definite integral and double integrals. In the case of the definite integral, the width of each subinterval was called $\Delta x$. As the number of subintervals went to $\infty$, the width of each subinterval went to 0. We called it $dx$. Similarly, for functions of two variables, the area of each subrectangle is $\Delta A = \Delta x \Delta y$. As the number of subrectangles approaches $\infty$, the area of each subrectangle approaches 0. We call it $dA$. You can think of $dA$ as being $dA = dx dy$, the area of each subrectangle as well as the length of their sides are approaching 0.

Remark 314  The definition of the integral does not require $f(x,y)$ to be positive on $R$. When $f(x,y) \geq 0$ on $R$, the integral is exactly the volume of the solid $S$.

Remark 315  We can approximate double integrals by using double Riemann sums. In the next section, we show a way to find the exact value of double integrals.
4.2 Iterated Integrals Over Rectangular Regions

Let \( f(x, y) \) be a function which is continuous on \( R = [a, b] \times [c, d] \). When we write \( \int_{c}^{d} f(x, y) \, dy \), we mean that \( x \) is held as a constant and we integrate with respect to \( y \). This is similar to partial differentiation with respect to \( y \), in the sense that \( x \) is held as a constant in both cases. This process is called partial integration with respect to \( y \).

**Example 316** Find \( \int_{1}^{2} x^2 \, dy \)

\[
\int_{1}^{2} x^2 \, dy = x^2 \int_{1}^{2} y \, dy \quad \text{since } x \text{ is a constant} \\
= x^2 \left[ \frac{y^2}{2} \right]_{1}^{2} \\
= x^2 \left( \frac{2 - 1}{2} \right) \\
= \frac{3}{2}x^2
\]

**Example 317** Find \( \int_{0}^{\pi} \sin x \sin y \, dy \)

\[
\int_{0}^{\pi} \sin x \sin y \, dy = \sin x \int_{0}^{\pi} \sin y \, dy \\
= \sin x \left[ -\cos y \right]_{0}^{\pi} \\
= \sin x \left[ -\cos \pi + \cos 0 \right] \\
= 2\sin x
\]

**Remark 318** When we compute an integral of the form \( \int_{c}^{d} f(x, y) \, dy \), we will be left with a function of \( x \). Call it \( A(x) \). In other words, \( A(x) = \int_{c}^{d} f(x, y) \, dy \).

We can now integrate \( A(x) \) with respect to \( x \) between \( a \) and \( b \). We get \( \int_{a}^{b} A(x) \, dx \). In fact, we have

\[
\int_{a}^{b} A(x) \, dx = \int_{a}^{b} \left[ \int_{c}^{d} f(x, y) \, dy \right] \, dx \quad (4.1)
\]

**Definition 319** An integral like the one on the right side of equation 4.1 is called an iterated integral.

**Remark 320** Usually, we omit the bracket, thus

\[
\int_{a}^{b} \left[ \int_{c}^{d} f(x, y) \, dy \right] \, dx = \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx \quad (4.2)
\]

To compute it, we integrate in two steps. First, we evaluate the integral inside, the one with respect to \( y \), holding \( x \) as a constant. This will give us a function of \( x \). We then integrate that function with respect to \( x \).
Remark 321 There is another iterated integral, obtained by switching the order of \( x \) and \( y \).

\[
\int_{c}^{d} \left[ \int_{a}^{b} f(x, y) \, dx \right] \, dy = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dxdy
\]

In this case, we first evaluate the inside integral, the one with respect to \( x \), holding \( y \) as a constant. It will give us a function of \( y \) which we then integrate with respect to \( y \).

Example 322 Evaluate \( \int_{0}^{\pi} \int_{0}^{\pi} \sin x \sin y \, dy \, dx \)

First, we evaluate \( \int_{0}^{\pi} \sin x \sin y \, dy \). In example 317, we found that it was \( 2 \sin x \).

Therefore,

\[
\int_{0}^{\pi} \int_{0}^{\pi} \sin x \sin y \, dy \, dx = \int_{0}^{\pi} 2 \sin x \, dx
\]

\[
= 2 \int_{0}^{\pi} \sin x \, dx
\]

\[
= 2 \left[ -\cos x \right]_{0}^{\pi}
\]

\[
= 2 \left( -\cos \frac{\pi}{2} + \cos 0 \right)
\]

\[
= 2
\]

Example 323 Evaluate \( \int_{0}^{\pi} \int_{0}^{\pi} \sin x \sin y \, dx \, dy \)

First, we evaluate

\[
\int_{0}^{\pi} \sin x \sin y \, dx = \sin y \int_{0}^{\pi} \sin x \, dx
\]

\[
= \sin y \left[ -\cos x \right]_{0}^{\pi}
\]

\[
= \sin y \left( -\cos \frac{\pi}{2} + \cos 0 \right)
\]

\[
= \sin y
\]

Therefore,

\[
\int_{0}^{\pi} \int_{0}^{\pi} \sin x \sin y \, dx \, dy = \int_{0}^{\pi} \sin y \, dy
\]

\[
= -\cos y \big|_{0}^{\pi}
\]

\[
= -\cos \pi + \cos 0
\]

\[
= 2
\]

Iterated integrals are important in evaluating double integrals, as the next theorem shows us. The theorem is stated without proof.
Theorem 324 (Fubini’s Theorem) Suppose that $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$. Then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

Remark 325 Note that the order of integration does not matter. We can first integrate with respect to $y$, then to $x$. We can also do the reverse. Make sure you use the correct limits of integration. For the $dx$ integral, the limits must be for the $x$–variable. For the $dy$ integral, the limits must be for the $y$–variable.

Example 326 Compute $\iint_R (x - 3y^2) \, dA$ where $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$.

Method 1: Using Fubini’s theorem, we get

$$\iint_R (x - 3y^2) \, dA = \int_0^2 \int_1^2 (x - 3y^2) \, dy \, dx$$

$$= \int_1^2 \left[ \frac{x^2}{2} - 3xy^2 \right]_0^2 \, dy$$

$$= \int_1^2 (2 - 6y^2) \, dy$$

$$= (2y - 2y^3)|_1^2$$

$$= (4 - 16) - (2 - 2)$$

$$= -12$$

Method 2: We can also use Fubini’s theorem, but with the opposite order of integration. We should get the same answer.

$$\iint_R (x - 3y^2) \, dA = \int_0^1 \int_0^2 (x - 3y^2) \, dx \, dy$$

$$= \int_0^2 \left[ xy - y^3 \right]_1^2 \, dx$$

$$= \int_0^2 [(2x - 8) - (x - 1)] \, dx$$

$$= \int_0^2 (x - 7) \, dx$$

$$= \left( \frac{x^2}{2} - 7x \right)|_0^2$$

$$= 2 - 14$$

$$= -12$$
Example 327 Find the volume of the solid bounded by \( z = f(x, y) = -x^2 - 2y^2 + 16 \), the planes \( x = 2, \ y = 2 \) and the three coordinate planes. It helps to try to sketch the function so we can see what the solid looks like. The graph is shown in figure 4.6. From the picture, we can see that the solid lies below the graph of \( z = f(x, y) = -x^2 - 2y^2 + 16 \), above the rectangle \( \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 0 \leq y \leq 2\} \). Therefore,

\[
\iiint_{R} f(x, y) \, dA = \int_{0}^{2} \int_{0}^{2} (-x^2 - 2y^2 + 16) \, dxdy
\]

\[
= \int_{0}^{2} \left[ \frac{-x^3}{3} - 2xy^2 + 16x \right]_{0}^{2} \, dy
\]

\[
= \int_{0}^{2} \left( \frac{-8}{3} - 4y^2 + 32 \right) \, dy
\]

\[
= \left( \frac{-8y^3}{3} - 4y^3 + 32y \right)_{0}^{2}
\]

\[
= \frac{-16}{3} - 32 + 64
\]

\[
= 48
\]

In some cases, the integral can be a little bit easier to evaluate. This happens when \( f \) can be written as the product of two functions, one a function of \( x \), the other one a function of \( y \). In this case, assuming \( f(x, y) = g(x) \cdot h(y) \), we have:

\[
\iiint_{R} f(x, y) \, dA = \int_{a}^{b} \int_{c}^{d} f(x, y) \, dydx \text{ by Fubini’s theorem}
\]

\[
= \int_{a}^{b} \int_{c}^{d} g(x) \cdot h(y) \, dydx
\]

\[
= \int_{a}^{b} g(x) \int_{c}^{d} h(y) \, dydx
\]

\[
= \int_{a}^{b} g(x) \, dx \int_{c}^{d} h(y) \, dy \text{ since } \int_{c}^{d} h(y) \, dy \text{ is a constant}
\]

So, we have the following proposition:

**Proposition 328** If \( f(x, y) = g(x) \cdot h(y) \), in other words if \( f(x, y) \) can be written as the product of two functions, one a function of \( x \), the other one a function of \( y \), then

\[
\iiint_{R} f(x, y) \, dA = \int_{a}^{b} g(x) \, dx \int_{c}^{d} h(y) \, dy
\]
Example 329 Evaluate $\int_0^{\pi} \int_0^{\pi} \sin x \sin y dy dx$

From the proposition, we have

$$\int_0^{\pi} \int_0^{\pi} \sin x \sin y dy dx = \int_0^{\pi} \sin x dx \int_0^{\pi} \sin y dy$$

$$= \left[ -\cos x \right]_0^{\pi} \left[ -\cos y \right]_0^{\pi}$$

$$= [1][2]$$

$$= 2$$

4.2.1 Summary

We list important points covered in this section.

- It is important to understand that in addition to finding antiderivative, the
  is an additional difficulty with multiple integrals: the region of integration.
  For the definite integral, that is the integral of a function of one variable,
  the region of integration was an interval. For multiple integrals, the region
  of integration will be a subset of the plane for functions of two variables,
  or a subset of space for functions of three variables. There are many more
  possibilities of such regions. This section focused on rectangular regions.
We learned to compute iterated integrals, that is integrals of the form 
\[ \int_a^b \int_c^d f(x,y) \, dy \, dx \] or 
\[ \int_c^d \int_a^b f(x,y) \, dx \, dy. \] Note there are two kinds, depending on the order of integration.

We learned to compute double integrals over a rectangular region, that is
integrals of the form 
\[ \iint_R f(x,y) \, dA \] where 
\[ R = [a,b] \times [c,d]. \]

A double integral is evaluated in terms of iterated integrals. More precisely, if 
\[ R = [a,b] \times [c,d], \] then Fubini’s theorem tells us that
\[ \iint_R f(x,y) \, dA = \int_a^b \int_c^d f(x,y) \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy. \]

Note that we can switch the order of integration when the region is a rectangle.

If the graph of \( z = f(x,y) \) is above the xy-plane then 
\[ \iint_R f(x,y) \, dA \] is the volume of the solid with cross section \( R \), between the xy-plane and \( z = f(x,y) \).

### 4.2.2 Assignment

Odd numbers 1-27 at the end of 12.1 in your book.