3.7 Maximum and Minimum Values

3.7.1 Introduction

In Calculus I (differential calculus for functions of one variable), the derivative was used successfully as a tool to find the maximum and minimum values of a function of one variable. In this section, we apply a similar concept to functions of two variables. The analogue of the derivative are the partials with respect to the variables of the function. We will see that partial derivatives play a role similar to that of the derivative. However, the situation is a little bit more complicated with functions of several variables. In order to understand this section better, the reader may want to review how maximum and minimum values are found in the case of functions of one variable.

3.7.2 Local Extreme Values

Given a function of two variables \( z = f(x, y) \), our goal is to find if it has local and global extreme values. If it does, we want to find where (at which points) these extreme values occur and what they are. We begin by giving more precise definitions of the concepts studied.

**Definition 289** Let \( f(x, y) \) be a function of two variables and \((a, b)\) a point in the domain of \( f \).

1. \( f \) has a **local maximum** at \((a, b)\) if \( f(x, y) \leq f(a, b) \) for every point \((x, y)\) near the point \((a, b)\). The number \( f(a, b) \) is called a local maximum.

2. If the above inequality holds for every point \((x, y)\) in the domain of \( f \), then \( f \) has an **absolute maximum** at \((a, b)\).

3. \( f \) has a **local minimum** at \((a, b)\) if \( f(x, y) \geq f(a, b) \) for every point \((x, y)\) near the point \((a, b)\). The number \( f(a, b) \) is called a local minimum.

4. If the above inequality holds for every point \((x, y)\) in the domain of \( f \), then \( f \) has an **absolute minimum** at \((a, b)\).

5. As in the one variable case, the local maxima and minima together comprise the **local extreme** values. The absolute maxima and minima together comprise the **absolute extreme** values.

To understand and visualize these concepts better, it helps to think of the graph of a function as a terrain. The local maxima are the peaks, the local minima are the valley bottoms.

As we did in calculus I, our approach is to find points which are candidates for local extreme values. We then test them to see which one really correspond to local extreme values. For functions of two variables, there is a theorem similar to Fermat’s theorem.
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Theorem 290 Let $f(x, y)$ be a function of two variables and $(a, b)$ a point in the domain of $f$. If $f$ has a local extremum at $(a, b)$ then either $\nabla f(a, b) = \mathbf{0}$ or $\nabla f(a, b)$ does not exist.

Proof. Let us assume that $f$ has a local extremum at $(a, b)$. We prove under the assumptions, if $\nabla f(a, b)$ exists, then we must have $\nabla f(a, b) = \mathbf{0}$. Saying that $\nabla f(a, b)$ exists means that both $\frac{\partial f(a, b)}{\partial x}$ and $\frac{\partial f(a, b)}{\partial y}$ must exist. Conversely, if one of the partials does not exist, then $\nabla f(a, b)$ will not exist. We define $g(x) = f(x, b)$. Then, $g'(a) = \frac{\partial f(a, b)}{\partial x}$. So, in particular, we see that the assumptions imply that $g'(a)$ exists. If $f$ has a local extremum at $(a, b)$, then $g$ has a local extreme at $x = a$. By Fermat’s theorem (functions of one variable version), since $g'(a)$ exists, it must be that $g'(a) = 0$. In other words, $\frac{\partial f(a, b)}{\partial x} = 0$.

Similarly, we define $G(y) = f(a, y)$. Then, $G'(b) = \frac{\partial f(a, b)}{\partial y}$. The assumptions imply that $G'(b)$ exists. If $G$ has a local extremum at $(a, b)$, then $G$ has a local extreme at $y = b$. By Fermat’s theorem, since $G'(b)$ exists, it must be that $G'(b) = 0$. In other words, $\frac{\partial f(a, b)}{\partial y} = 0$. We have proven that under the assumptions, both $\frac{\partial f(a, b)}{\partial x} = 0$ and $\frac{\partial f(a, b)}{\partial y} = 0$. This means that $\nabla f(a, b) = \mathbf{0}$.

Remark 291 The theorem could have been stated differently. Instead of saying $\nabla f(a, b) = \mathbf{0}$, we could have said $\frac{\partial f(a, b)}{\partial x} = 0$ and $\frac{\partial f(a, b)}{\partial y} = 0$ because for a vector to be the zero vector, all its components must be 0. You will also note that for $\nabla f(a, b)$ not to exist, it is enough for one of its component not to exist that is one of the partials at $(a, b)$ not to exist.

Remark 292 Geometrically, the theorem says that if $f$ has a tangent plane at $(a, b)$ and $f$ has a local extremum at $(a, b)$, then the tangent plane must be horizontal. You will see that this is similar to the case of functions of one variable. In the case of functions of one variable, Fermat’s theorem says that if a function has a tangent line and an extremum at a point, then the tangent line must be horizontal at that point.

Remark 293 Unfortunately, like in the one dimensional case, the reverse is not true. A point $(a, b)$ at which either $\nabla f(a, b) = \mathbf{0}$ or $\nabla f(a, b)$ does not exist, does not necessarily correspond to an extreme value. These points will simply be candidates. We will need to test these candidates to see which ones correspond to extreme values. Because such points play such an important role, we give them a special name.

Definition 294 A point $(a, b)$ is called a critical point of the function $f(x, y)$ if the following conditions are satisfied:
1. \((a, b)\) is an interior point of the domain of \(f\).

2. Either \(\nabla f (a, b) = \overrightarrow{0}\) or \(\nabla f (a, b)\) does not exist.

Remark 295 Theorem 290 tells us that a local extreme value can only occur at a critical point. However, not every critical point corresponds to an extreme value.

Example 296 Find the critical points of \(f (x, y) = y^2 - xy + 2x + y + 1\). Clearly, \(f\) is defined everywhere. To find the critical points, we first find the first partials of \(f\).

\[
\frac{\partial f (x, y)}{\partial x} = -y + 2
\]

and

\[
\frac{\partial f (x, y)}{\partial y} = 2y - x + 1
\]

The partials of \(f\) are polynomials, so they are always defined. Also, \(\nabla f (x, y) = 0\) if and only if \(\frac{\partial f (x, y)}{\partial x} = 0\) and \(\frac{\partial f (x, y)}{\partial y} = 0\). Now, \(\frac{\partial f (x, y)}{\partial x} = 0\) if and only if \(y = 2\) and \(\frac{\partial f (x, y)}{\partial y} = 0\) if and only if \(x = 2y + 1\). Combining the two gives \(y = 2\) and \(x = 5\). So we see that \(\nabla f (x, y) = 0\) at \((5, 2)\), and this is the only critical point. We do not yet know that it corresponds to an extreme value. We can only conclude that \(f\) has at most one extreme value.

Example 297 Find the critical points of \(f (x, y) = 1 + \sqrt{x^2 + y^2}\)

\[
\frac{\partial f (x, y)}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}
\]

and

\[
\frac{\partial f (x, y)}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}
\]

We see that \(\nabla f (x, y)\) is not defined at \((0, 0)\), so \((0, 0)\) is a critical point. \(\nabla f (x, y)\) is never \(\overrightarrow{0}\) (why?). So, \((0, 0)\) is the only critical point.

Before we learn how to test critical points to see which ones correspond to local extreme values, let us see examples which illustrate why not every critical point corresponds to an extreme value.

Example 298 Remember that for functions of one variables, functions such as \(y = x^3\) have a horizontal tangent yet no local extreme value. There are similar examples in the case of functions of two variables. Consider \(f (x, y) = x^3\).

Clearly, \(\nabla f (0, 0) = \overrightarrow{0}\). Yet, \(f\) does not have a local extreme value at \((0, 0)\) (see figure 3.17). In fact, \(\nabla f (0, y) = \overrightarrow{0}\) for any \(y\), yet \(f\) does have a local extreme value along the \(y\)-axis (see figure 3.17).
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Example 299 A case which does not occur with functions of one variable is when we have a saddle. You will recall that one of the tests, the second derivative test, says that if \( f(x) \) is such that \( f'(a) = 0 \) then \( f \) has a local maximum at \( x = a \) if \( f''(a) < 0 \) (\( f \) concave down) and a local minimum at \( x = a \) if \( f''(a) > 0 \) (\( f \) concave up). For a function \( f(x,y) \), the situation is more complex. There are two directions to consider. It is possible for \( f \) to be concave up at a point \((a, b)\) in the \( x \) direction and concave down at \((a, b)\) in the \( y \) direction. In this case, even if \( \nabla f(a, b) = 0 \), the point \((a, b)\) corresponds neither to a maximum nor a minimum. Such a point \((a, b)\) is called a saddle point. Figure 3.18 illustrates this concept. \((0, 0)\) is a saddle point. The graph is concave up in the \( x \) direction at \((0, 0)\) and concave down in the \( y \) direction.

We now state the equivalent of the second derivative test, for functions of two variables. This test is a little bit more complex than the second derivative test and its proof far more complicated. We will give the test without proof. A proof can be found in most advanced calculus books.

Theorem 300 Let \( f \) be a function of two variables and \((a, b)\) a point in the domain of \( f \). Suppose that the second partials of \( f \) are continuous on a disk centered at \((a, b)\) and \( \nabla f(a, b) = \vec{0} \) (that is \((a, b)\) is a critical point of \( f \), also
Figure 3.18: Graph of a saddle. (0, 0) is a saddle point
the graph of \( z = f(x, y) \) has a horizontal tangent plane at \((a, b)\). Let

\[
D = \left( \frac{\partial^2 f}{\partial x^2}(a, b) \frac{\partial^2 f}{\partial y^2}(a, b) \right)^2 - \left( \frac{\partial^2 f}{\partial x \partial y}(a, b) \right)^2
\]

1. If \( D > 0 \) and \( \frac{\partial^2 f}{\partial x^2}(a, b) > 0 \), then \( f(a, b) \) is a local minimum.
2. If \( D > 0 \) and \( \frac{\partial^2 f}{\partial x^2}(a, b) < 0 \), then \( f(a, b) \) is a local maximum.
3. If \( D < 0 \), then \( f(a, b) \) is not a local maximum or minimum.

**Remark 301** In the last case, when \( D < 0 \), \((a, b)\) is called a saddle point.

**Remark 302** If \( D = 0 \), the test provides no information.

**Remark 303** To remember the formula for \( D \), write it as the determinant

\[
D = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}
\]

**Example 304** Find the local extreme values of \( f(x, y) = 2x^2 + y^2 - xy - 7y \).

We must first find the critical numbers. For this, we compute the first partials of \( f \).

\[
\frac{\partial f}{\partial x}(x, y) = 4x - y \\
\frac{\partial f}{\partial y}(x, y) = 2y - x - 7
\]

These partials are always defined. To find when they are 0, we must solve

\[
\begin{cases} 
4x - y = 0 \\
2y - x - 7 = 0
\end{cases}
\]

The solution is: \( \{x = 1, y = 4\} \). We must now test this critical point. For this, we begin by computing the second partials of \( f \).

\[
\frac{\partial^2 f}{\partial x^2}(x, y) = 4 \\
\frac{\partial^2 f}{\partial y^2}(x, y) = 2 \\
\frac{\partial^2 f}{\partial x \partial y}(x, y) = -1
\]

So,

\[
D = \frac{\partial^2 f}{\partial x^2}(1, 4) \frac{\partial^2 f}{\partial y^2}(1, 4) - \left( \frac{\partial^2 f}{\partial x \partial y}(1, 4) \right)^2 \\
= (4)(2) - (-1)^2 \\
= 7
\]

Since both \( D \) and \( \frac{\partial^2 f}{\partial x^2}(1, 4) \) are positive, it follows that \( f(1, 4) \) is a local minimum for \( f \). \( f(1, 4) = -14 \). The graph of \( f \) is shown in figure 3.19.
Example 305 Find the local extreme values of \( f(x, y) = x^3 + y^3 - 3xy \)

First, we find the critical points, that is points where \( \nabla f \) is either 0 or undefined. For this, we compute the partials of \( f \).

\[
\frac{\partial f}{\partial x}(x, y) = 3x^2 - 3y \\
\frac{\partial f}{\partial y}(x, y) = 3y^2 - 3x
\]

The partials are always defined. They are 0 when

\[
\begin{cases} 
  x^2 - y = 0 \\
  y^2 - x = 0
\end{cases}
\]

Solving for \( y \) in the first equation and replacing in the second equation gives

\[
\begin{align*}
x^4 - x &= 0 \\
x(x^3 - 1) &= 0 \\
x &= 0 \text{ or } x = 1
\end{align*}
\]

Since \( y = x^2 \) (first equation), we see that when \( x = 0 \), \( y = 0 \) and when \( x = 1 \), \( y = 1 \). So, there are two critical points. They are \((0,0)\) and \((1,1)\). To test these
critical points, we need to compute the second partials of $f$.

$$\frac{\partial^2 f}{\partial x^2} (x,y) = 6x$$

$$\frac{\partial^2 f}{\partial y^2} (x,y) = 6y$$

$$\frac{\partial^2 f}{\partial x \partial y} (x,y) = -3$$

We evaluate these partials at the critical points. When there are several points to consider, this is best done using a table such as the one below.

<table>
<thead>
<tr>
<th>Point $(a, b)$</th>
<th>$\frac{\partial^2 f}{\partial x^2} (a, b)$</th>
<th>$\frac{\partial^2 f}{\partial y^2} (a, b)$</th>
<th>$\frac{\partial^2 f}{\partial x \partial y} (a, b)$</th>
<th>D</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0)$</td>
<td>0</td>
<td>0</td>
<td>$-3$</td>
<td>$-9$</td>
<td>Saddle Point</td>
</tr>
<tr>
<td>$(1, 1)$</td>
<td>6</td>
<td>6</td>
<td>$-3$</td>
<td>$27$</td>
<td>Local Minimum</td>
</tr>
</tbody>
</table>

So, we see that $f(1, 1) = -1$ is a local minimum. $f(0, 0) = 0$ is a saddle point, it is not a local extreme value. The graph of $f$ is shown in figure 3.20.

**Example 306** Find the local extreme values of $f(x, y) = -xye^{-\frac{x^2+y^2}{2}}$
We proceed as above.

\[
\frac{\partial f}{\partial x}(x, y) = -ye^{-\frac{x^2+y^2}{2}} + x^2ye^{-\frac{x^2+y^2}{2}} = y(x^2 - 1)e^{-\frac{x^2+y^2}{2}}
\]

\[
\frac{\partial f}{\partial y}(x, y) = -xe^{-\frac{x^2+y^2}{2}} + xy^2e^{-\frac{x^2+y^2}{2}} = x(y^2 - 1)e^{-\frac{x^2+y^2}{2}}
\]

The partials are always defined. They are 0 when \(y(x^2 - 1) = 0\) and \(x(y^2 - 1) = 0\).

The first equations gives us \(y = 0\) (\(x\) anything) or \(x = \pm 1\) (\(y\) anything). When \(y = 0\), the second equation implies that \(x = 0\). So, \((0, 0)\) is a critical point.

When \(x = 1\), the second equation gives us \(y = \pm 1\). Next, we compute the second partials of \(f\) and evaluate them at the critical points.

\[
\frac{\partial^2 f}{\partial x^2}(x, y) = xy(3 - x^2)e^{-\frac{x^2+y^2}{2}}
\]

\[
\frac{\partial^2 f}{\partial y^2}(x, y) = xy(3 - y^2)e^{-\frac{x^2+y^2}{2}}
\]

\[
\frac{\partial^2 f}{\partial x \partial y}(x, y) = (x^2 - 1)(1 - y^2)e^{-\frac{x^2+y^2}{2}}
\]

<table>
<thead>
<tr>
<th>Point ((a, b))</th>
<th>(\frac{\partial^2 f}{\partial x^2}(a, b))</th>
<th>(\frac{\partial^2 f}{\partial y^2}(a, b))</th>
<th>(\frac{\partial^2 f}{\partial x \partial y}(a, b))</th>
<th>D</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(-1)</td>
<td>Saddle Point</td>
</tr>
<tr>
<td>((1, 1))</td>
<td>(\frac{1}{e})</td>
<td>(\frac{1}{e})</td>
<td>(\frac{1}{e})</td>
<td>(\frac{1}{e})</td>
<td>Local Minimum</td>
</tr>
<tr>
<td>((1, -1))</td>
<td>(-\frac{1}{e})</td>
<td>(-\frac{1}{e})</td>
<td>0</td>
<td>(\frac{1}{e})</td>
<td>Local Maximum</td>
</tr>
<tr>
<td>((-1, 1))</td>
<td>(-\frac{1}{e})</td>
<td>(-\frac{1}{e})</td>
<td>0</td>
<td>(\frac{1}{e})</td>
<td>Local Maximum</td>
</tr>
<tr>
<td>((-1, -1))</td>
<td>(\frac{1}{e})</td>
<td>(\frac{1}{e})</td>
<td>0</td>
<td>(\frac{1}{e})</td>
<td>Local Minimum</td>
</tr>
</tbody>
</table>

Therefore, we see that \(f(0, 0) = 0\) is a saddle point, \(f(1, 1) = \frac{1}{e}\) and \(f(-1, -1) = \frac{1}{e}\) are local minima, \(f(1, -1) = \frac{1}{e}\) and \(f(-1, 1) = \frac{1}{e}\) are local maxima. The graph of \(f\) is shown in figure 3.21.

### 3.7.3 Absolute Extreme Values

For functions of one variable, the Extreme Value Theorem says that if such functions are continuous on a closed interval, then they have an absolute maximum and minimum on that interval. These absolute extrema can occur at a critical point but also at the endpoints of the interval. This result can be extended to functions of two or more variables. In the case of functions of two variables, the term "end point of the interval" will be replaced by "boundary of the set".
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Theorem 307 (Extreme Value Theorem (functions of two variables))
If \( f \) is continuous on a closed and bounded set \( D \) in \( \mathbb{R}^2 \), then \( f \) has both an absolute maximum and a minimum on \( D \). That is there exists points \((a, b)\) and \((c, d)\) in \( D \) such that \( f(a, b) \) is an absolute maximum and \( f(c, d) \) is an absolute minimum.

Absolute extrema can happen at critical points of \( f \) in \( D \). They can also happen on the boundary of \( D \). Unfortunately, finding the extreme values of \( f \) on the boundary is in general difficult and requires special methods we will not cover. We can do it in very simple cases as the next example illustrates. First, let us summarize the steps to follow in order to find the local extrema of a function \( f \) on a closed and bounded set \( D \).

Proposition 308 To find the local extrema of a continuous function \( f \) on a closed and bounded set \( D \):

1. Find the critical points of \( f \) and the values of \( f \) at these critical points.

2. Find the extreme values of \( f \) on the boundary of \( D \).

3. The largest value from steps 1 and 2 is the absolute maximum, the smallest is the absolute minimum.
Example 309 Find the absolute extrema of $f(x, y) = x^2 - 2xy + 2y$ on $D$ where $D = \{(x, y) \mid 0 \leq x \leq 3 \text{ and } 0 \leq y \leq 2\}$. 

We begin by finding the critical points.

$$\frac{\partial f}{\partial x}(x, y) = 2x - 2y$$

and

$$\frac{\partial f}{\partial y}(x, y) = -2x + 2$$

These partials are always defined. They are 0 when

$$\begin{cases} x - y = 0 \\ x - 1 = 0 \end{cases}$$

Solving these two equations simultaneously gives $x = 1$ and $y = 1$. So, $(1, 1)$ is a critical point. $f(1, 1) = 1$. Next, we find the extreme values of $f$ on the boundary of $D$. Note that $D$ is a rectangle. Its boundary consists of 4 line segments. The vertical line $x = 0$, between $(0, 0)$ and $(0, 2)$, the vertical line $x = 3$, between $(3, 0)$ and $(3, 2)$, the horizontal line $y = 0$, between $(0, 0)$ and $(3, 0)$ and the horizontal line $y = 2$, between $(0, 2)$ and $(3, 2)$.

- Along the line $x = 0$, with $0 \leq y \leq 2$, $f(x, y) = f(0, y) = 2y$. This is an increasing function of $y$. Since this line segment is between $(0, 0)$ and $(0, 2)$, its largest value happens at $(0, 2)$ and it is $f(0, 2) = 4$. Its smallest value happens at $(0, 0)$ and it is $f(0, 0) = 0$.

- Along the line $x = 3$, with $0 \leq y \leq 2$, $f(x, y) = f(3, y) = 9 - 6y + 2y = -4y + 9$. This is a decreasing function of $y$. Since this line segment is between $(3, 0)$ and $(3, 2)$, its largest value happens at $(3, 0)$ and it is $f(3, 0) = 9$. Its smallest value happens at $(3, 2)$ and it is $f(3, 2) = 1$.

- Along the line $y = 0$, with $0 \leq x \leq 3$, $f(x, y) = f(x, 0) = x^2$. This is an increasing function of $x$ when $0 \leq x \leq 3$. Its largest value is $f(3, 0) = 9$, its smallest value is $f(0, 0) = 0$.

- Along the line $y = 2$, with $0 \leq x \leq 3$, $f(x, y) = f(x, 2) = x^2 - 4x + 4$. We can use the methods of calculus I to find the extreme values of this function. In this case, we can also notice that $f(x, 2) = (x - 2)^2$. So, it is minimum at $x = 2$, the minimum is $f(2, 2) = 0$. The maximum happens at 0 and it is $f(0, 2) = 4$.

- To summarize, we found the following:

<table>
<thead>
<tr>
<th>Point $(a, b)$</th>
<th>$(1, 1)$</th>
<th>$(0, 2)$</th>
<th>$(0, 0)$</th>
<th>$(3, 0)$</th>
<th>$(3, 2)$</th>
<th>$(2, 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(a, b)$</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>9</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

In conclusion, $f(3, 0) = 9$ is the absolute maximum, $f(0, 0) = f(2, 2) = 0$ is the absolute minimum. The graph of $f$ is shown in figure 3.22.

### 3.7.4 Assignment

Do odd# 1-25, 31, 33 at the end of 11.7 in your book.
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Figure 3.22: $f(x, y) = x^2 - 2xy + 2y$