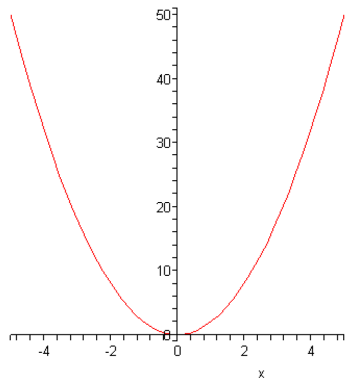

Figure 1.18: Parabola $y = 2x^2$

1.6 Quadric Surfaces

1.6.1 Brief review of Conic Sections

You may need to review conic sections for this to make more sense. You should know what a parabola, hyperbola and ellipse are. We remind you of their equation here. If you have never seen these equations before, consult a calculus book to study them in more detail.

- Parabola: $y = ax^2$ or $x = ay^2$. The parabola is along the axis corresponding to the variable not being squared. Figures 1.18 and 1.19 illustrate this.

- Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. An ellipse looks like an elongated circle. It extends in the $x - direction$ from $-a$ to a and in the $y - direction$ from $-b$ to b. Figure 1.20 illustrates this.

- Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. The hyperbola never crosses the axis corresponding to the variable having the negative sign. Figures 1.21 and 1.22 illustrate this.

1.6.2 Introduction to Quadric Surfaces

A quadric surface is the graph of a second degree equation in three variables. The general form of such an equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

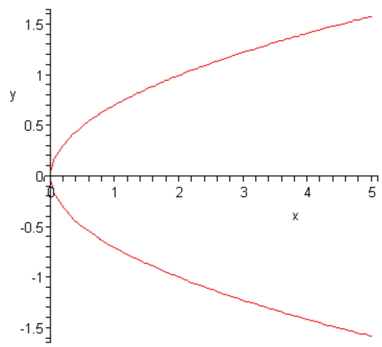

Figure 1.19: Parabola $x = 2y^2$

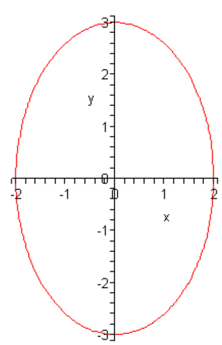

Figure 1.20: Ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

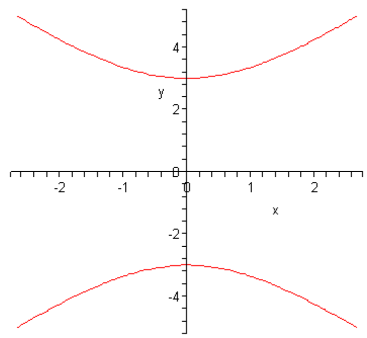

Figure 1.21: Hyperbola $\dfrac{-x^2}{4} + \dfrac{y^2}{9} = 1$

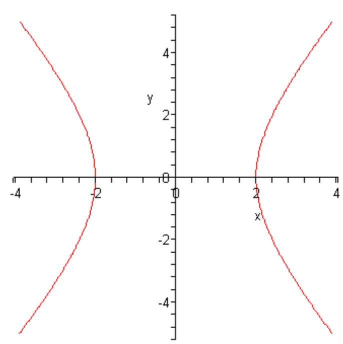

Figure 1.22: Hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

where A, B, ...,J are constants. Quadric surfaces can be classified into five categories: ellipsoids, hyperboloids, cones, paraboloids, and quadric cylinders. In this document, we will study these categories and learn how to identify them. Quadrics surfaces are the 3-D equivalent of conic sections in the plane. You should review conic sections if you are not familiar with them.

For each category, we will learn to find elements which characterize it. These elements include:

1. Standard equation
2. Traces. They are the curves of intersection of the surface with planes parallel to the coordinate planes.
3. Intercepts. They are the points of intersection of the surface with the coordinate axes.

1.6.3 Classification of Quadric Surfaces

Quadric surfaces fall into nine categories. To know which one, one has to rewrite it so that it has one of the forms below. Rewriting a quadric surface often involves completing the square. Make sure you remember how to do this.

Ellipsoid

It is the 3-D equivalent of an ellipse in 2-D. Its standard equation is of the form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

The traces of an ellipsoid are ellipses. This is easy to see. For example, if we try to find the intersection of an ellipsoid with the plane parallel to the $xy-$plane, that is the plane with equation $z=k$ for some number k, we obtain:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2}$$

which is an ellipse as long as $-c<k<c$. If $k>c$ or $k<-c$, the two do not intersect.

The intercepts of the ellipsoid are $(\pm a,0,0)$, $(0,\pm b,0)$, $(0,0,\pm c)$.

An example of an ellipsoid is shown in Figure 1.23

Hyperboloid

There are two types depending on how many pieces the surface has.

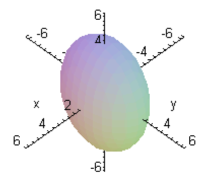

Figure 1.23: Graph of $\dfrac{x^2}{4}+\dfrac{y^2}{9}+\dfrac{z^2}{16}=1$

Hyperboloid of one sheet The standard equation is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

The minus sign could also be in front of the first or second fraction. However, there can only be **one** minus sign.

In the example shown in Figure 1.24, the minus sign is in front of the z variable. The axis of the paraboloid is the $z-axis$. The intersection of the paraboloid with planes perpendicular to the axis of the hyperboloid (the $xy-plane$ in this case) are ellipses. The intersection with planes parallel to the axis of the hyperboloid are hyperbolas. If, for example we put the minus sign in front of the x variable, then the axis of the hyperboloid will be the $x-axis$. This time, it is the intersection with planes parallel to the $yz-plane$ which will be ellipses. The graph of such an example is shown in Figure 1.25

In general, the axis of a hyperboloid of one sheet is the axis corresponding to the variable having the minus sign.To see that the intersection of the hyperboloid shown in figure 1.24 with a plane parallel to the $xy-plane$ is an ellipse, we let $z=k$ in its equation. We get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}$$

which is an ellipse. To see that the intersection of the hyperboloid shown in figure 1.24 with a plane parallel to the $yz-plane$ is an hyperbola, we let $x=k$ in its equation. We get

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1-\frac{k^2}{a^2}$$

which is an hyperbola.

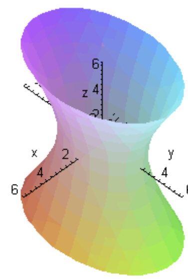

Figure 1.24: Hyperboloid of one sheet $\dfrac{x^2}{4} + \dfrac{y^2}{9} - \dfrac{z^2}{16} = 1$

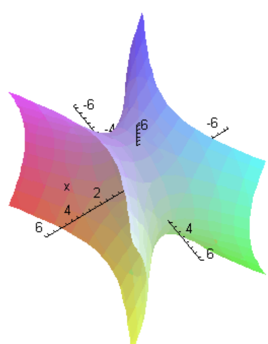

Figure 1.25: Hyperboloid of one sheet $\dfrac{-x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$

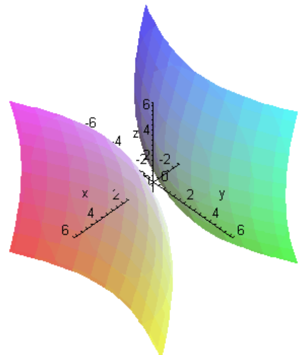

Figure 1.26: Hyperboloid of two sheets $\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{16} = 1$

Hyperboloid of two sheets The standard equation is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

The minus signs could be in front of other fractions also. There has to be **two** minus signs to have a hyperboloid of two sheets. The number of sheets is the same as the number of minus signs. The axis of a hyperboloid of two sheets is the axis corresponding to the variable not having a minus sign. The intersection of an hyperboloid of two sheets with a plane parallel to its axis is a hyperbola. Its intersection with a plane perpendicular to its axis is an ellipse when it exists. Figure 1.26 shows a hyperboloid of two sheets with axis the $x - axis$. Figure 1.27 shows a hyperboloid of two sheets with axis the $y - axis$.

Elliptic Cones

The standard equation is the same as for a hyperboloid, replacing the 1 on the right side of the equation by a 0. Whether we have one minus sign or two, we get an equation of the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

The axis of the cone corresponds to the variable on the right side of the equation.

Figure 1.28 shows a cone with axis the $z - axis$. Figure 1.29 shows a cone with axis the $y - axis$.

Due to the limitations of the graphing software used for this document, it may appear that the graphs shown in Figure 1.28 and Figure 1.29 are made of two pieces. It is not so. If you look at their equations, you will see that they both contain the point $(0, 0, 0)$. Hence, they should be connected. The

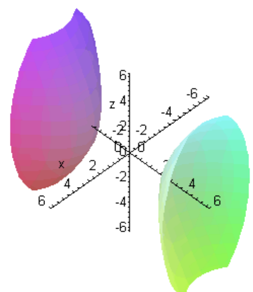

Figure 1.27: Hyperboloid of two sheets $\frac{-x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$

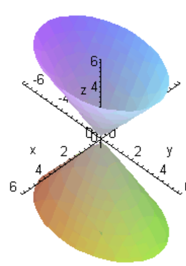

Figure 1.28: The cone $\frac{x^2}{4} + \frac{y^2}{9} = \frac{z^2}{16}$

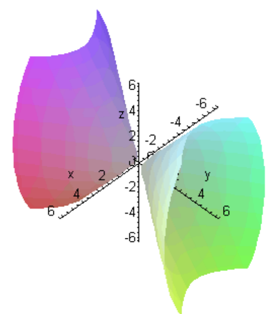

Figure 1.29: The cone $\dfrac{x^2}{4}+\dfrac{z^2}{9}=\dfrac{y^2}{16}$

intersection of a cone with a plane perpendicular to its axis is an ellipse. The intersection of a cone with a plane parallel to its axis is either a hyperbola if the plane does not contain the origin or a pair of lines if the plane contains the origin.

The surface of a cone has the property that if P is any point on the cone, then the line segment OP lies entirely on the cone.

Paraboloid

If we start with the equation of a cone, and remove the squares to the expression on the right side of the equal sign, we obtain a paraboloid., There are two types of paraboloids, depending on whether the expression to the left of the equal sign has a minus sign or not.

Elliptic paraboloid The standard equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$$

Figure 1.30 shows a paraboloid with axis the $z-axis$. The intersection it makes with a plane perpendicular to its axis is an ellipse. The other traces are parabolas. If we change the sign of c, the paraboloid is oriented the other way as shown in Figure 1.31. The axis along which the paraboloid extends corresponds to the variable not being squared.

Hyperbolic paraboloid The standard equation is:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$$

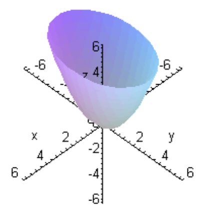

Figure 1.30: Elliptic paraboloid $\dfrac{x^2}{4} + \dfrac{y^2}{9} = \dfrac{z}{4}$

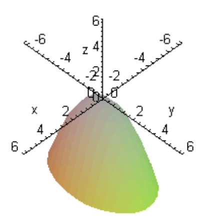

Figure 1.31: Elliptic paraboloid $\dfrac{x^2}{4} + \dfrac{y^2}{9} = \dfrac{-z}{4}$

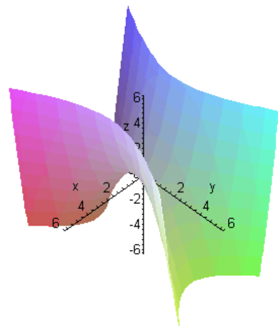

Figure 1.32: Hyperbolic paraboloid $\frac{x^2}{4} - \frac{y^2}{2} = \frac{z}{4}$

An example is shown in Figure 1.32. This graph is also called a saddle. We will visit it again when we study the problem of finding extrema of functions of several variables. For this example, the intersection with a plane parallel to the $xy-plane$ is a hyperbola. The intersection with the other coordinate planes is a parabola.

Quadric cylinders

There are three types.

Elliptic cylinder The standard equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Figure 1.33 shows such a cylinder. Its axis is the $z-axis$, the axis corresponding to the variable missing in the equation. Its intersection with the $xy-plane$, the plane containing the two variables present in the equation, is an ellipse.

Hyperbolic cylinder The standard equation is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Figure 1.34 shows such a cylinder. Its axis is the $z-axis$, the axis corresponding to the variable missing in the equation. Its intersection with the $xy-plane$, the plane containing the two variables present in the equation, is a hyperbola.

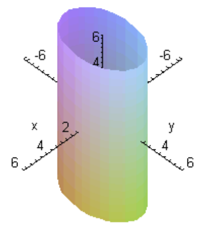

Figure 1.33: The elliptic cylinder $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

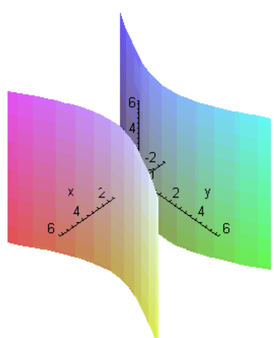

Figure 1.34: The hyperbolic cylinder $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

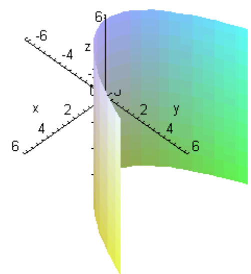

Figure 1.35: Parabolic cylinder $y = \dfrac{x^2}{4}$

Parabolic cylinder The standard equation is

$$y = ax^2$$

Figure 1.35 shows such a cylinder. It appears to be wrapped around the axis corresponding to the missing variable. Its intersection with the $xy - plane$, the plane containing the two variables present in the equation, is a parabola.

Conclusion

We have studied the following forms:

Name	**Equation**
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
Hyperboloid of two sheets	$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
Elliptic cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$
Elliptic paraboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$
Hyperbolic paraboloid	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$
Elliptic cylinder	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Hyperbolic cylinder	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Parabolic cylinder	$y = ax^2$

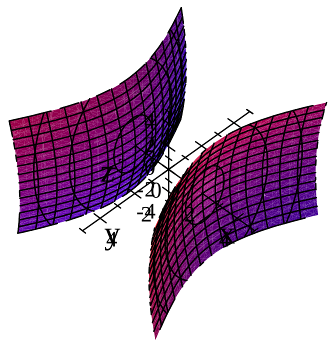

Figure 1.36: Example 1

Classification - Examples

Classifying a quadric surface amounts to transforming its equation until it has one of the forms described above. This often involves "completing the square".

In all the above equations, it is possible to have $(x-h)^2$ instead of x^2, or $(y-k)^2$ instead of y^2 or $(z-l)^2$ instead of z^2. You will recall from pre-calculus that this will simply translate the object, but will not change its shape. For example $\frac{(x-1)^2}{2}+\frac{(y+2)^2}{3}+\frac{z^2}{4}=1$ has the same shape as $\frac{x}{2}+\frac{y}{3}+\frac{z^2}{4}=1$. It is translated from it 1 unit in the positive x direction and 2 units in the negative y direction.

Example 1 *Classify*

$$4x^2-9y^2+z^2+36=0$$

We only have terms in x^2, y^2, and z^2. We won't have to complete the square. We simply group these terms together, and put everything else to the right of the equal sign. We obtain:

$$4x^2-9y^2+z^2=-36$$

Since we want a 1 on the right, we divide by -36, we obtain

$$-\frac{x^2}{9}+\frac{y^2}{4}-\frac{z^2}{36}=1$$

This is a hyperboloid of two sheets. Its graph is shown in Figure 1.36.Its intersection with a plane parallel to the $xy-plane$, that is a plane of coordinates $z=k$ is

$$-\frac{x^2}{9}+\frac{y^2}{4}=1+\frac{k^2}{32}$$

which is a hyperbola. As an exercise, find the other traces.

Example 2 *Classify*

$$x^2 + 4y^2 + z^2 - 2x = 0$$

We have a term in x^2 and x, so we will have to complete the square.

$$\begin{aligned} x^2 - 2x + 4y^2 + z^2 &= 0 \\ x^2 - 2x + 1 + 4y^2 + z^2 &= 1 \\ (x-1)^2 + \frac{y^2}{\left(\frac{1}{2}\right)^2} + z^2 &= 1 \end{aligned}$$

This is an ellipsoid centered at $(1, 0, 0)$. Its intercepts are $(0, 0, 0)$, $(2, 0, 0)$.

Make sure you can do the above before attempting the problems.

1.6.4 Problems

Do # 1-25 (odd numbers) at the end of 9.6 in your book.