We extend the notion of limits studied in Calculus I. Recall that when we write \( \lim_{x \to a} f(x) = L \), we mean that \( f \) can be made as close as we want to \( L \), by taking \( x \) close enough to \( a \) but not equal to \( a \). In this process, \( f \) has to be defined near \( a \), but not necessarily at \( a \). The information we are trying to derive is the behavior of \( f(x) \) as \( x \) gets closer to \( a \). When we extend this notion to functions of two variables (or more), we will see that there are many similarities. We will discuss these similarities. However, there is also a main difference. The domain of functions of two variables is a subset of \( \mathbb{R}^2 \), in other words it is a set of pairs. A point in \( \mathbb{R}^2 \) is of the form \((x, y)\). So, the equivalent of \( x \to a \) will be \((x, y) \to (a, b)\). For functions of three variables, the equivalent of \( x \to a \) will be \((x, y, z) \to (a, b, c)\), and so on. One important difference is that while \( x \) could only approach \( a \) from two directions, from the left or from the right, \((x, y)\) can approach \((a, b)\) from infinitely many directions. In fact, it does not even have to approach \((a, b)\) along a straight path as shown in the next slide.
Introduction

With functions of one variable, one way to show a limit existed, was to show that the limit from both directions existed and were equal \( \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) \). Equivalently, when the limits from the two directions were not equal, we concluded that the limit did not exist. For functions of several variables, we would have to show that the limit along every possible path exist and are the same. The problem is that there are infinitely many such paths. To show a limit does not exist, it is still enough to find two paths along which the limits are not equal. In view of the number of possible paths, it is not always easy to know which paths to try. We give some suggestions here. You can try the following paths:

1. Horizontal line through \((a, b)\), the equation of such a path is \(y = b\).
2. Vertical line through \((a, b)\), the equation of such a path is \(x = a\).
3. Any straight line through \((a, b)\), the equation of the line with slope \(m\) through \((a, b)\) is \(y = mx + b - am\).
4. Quadratic paths. For example, a typical quadratic path through \((0, 0)\) is \(y = x^2\).
While it is important to know how to compute limits, it is also important to understand what we are trying to accomplish. Like for functions of one variable, when we compute the limit of a function of several variables at a point, we are simply trying to study the behavior of that function near that point. The questions we are trying to answer are:

1. Does the function behave nicely near the point in question? In other words, does the function seem to be approaching a single value as its input is approaching the point in question?

2. Is the function getting arbitrarily large (going to $\infty$ or $-\infty$)?

3. Does the function behave erratically, that is it does not seem to be approaching any value?

In the first case, we will say that the limit exists and is equal to the value the function seems to be approaching. In the other cases, we will say that the limit does not exist.
Definitions

**Definition**

We write \( \lim_{(x,y) \to (a,b)} f(x,y) = L \) and we read the limit of \( f(x,y) \) as \((x,y)\) approaches \((a,b)\) is \(L\), if we can make \(f(x,y)\) as close as we want to \(L\), simply by taking \((x,y)\) close enough to \((a,b)\) but not equal to it.

Let us make a few remarks.

1. When computing \( \lim_{(x,y) \to (a,b)} f(x,y) \), \((x,y)\) is never equal to \((a,b)\). In fact, \(f\) may not even be defined at \((a,b)\). However \(f\) must be defined at the points \((x,y)\) we consider as \((x,y) \to (a,b)\).

2. There are several notation for this limit. They all represent the same thing, we list them below.
   - \( \lim_{(x,y) \to (a,b)} f(x,y) = L \)
   - \( \lim_{x \to a} f(x,y) = L \)
   - \( \lim_{y \to b} f(x,y) = L \)
   - \( f(x,y) \) approaches \(L\) as \((x,y)\) approaches \((a,b)\).
I will not discuss this method extensively and you will not be tested on it. I will simply illustrate it by looking at one example. It is the equivalent of the technique used in calculus I to evaluate $\lim_{x \to a} f(x)$ by building a table of values for $f(x)$ with values of $x$ closer and closer to $a$. For functions of two variables, we have to evaluate $\lim_{(x,y) \to (a,b)} f(x,y)$. We try to estimate or guess if a limit exists and what its value is by looking at a table of values. Such a table will be more complicated than in the case of functions of one variable. When $(x, y) \to (a, b)$, we have to consider all possible combinations of $x \to a$ and $y \to b$. This usually results in a square table as the one shown on the next slide. As we will see, the table is more useful to determine if a limit does not exist by identifying two paths along which the limit has different values.

We will compute $\lim_{(x,y) \to (0,0)} f(x,y)$ for $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$. 
The table can be used to estimate the limit along various paths

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Main diagonal: path $y = x$
Along $y = 0$, it seems the limit is 1

Other diagonal: path $y = -x$
Along $x = 0$, it seems the limit is -1

Column: path $y =$ constant
So, it seems the limit does not exist.

Row: path $x =$ constant
Computing limits using the analytical method is computing limits using the limit rules and theorems. We will see that these rules and theorems are similar to those used with functions of one variable. We present them without proof, and illustrate them with examples.

We list these properties for functions of two variables. Similar properties hold for functions of more variables. Let us assume that $L$, $M$, $c$ and $k$ are real numbers and that $\lim_{(x,y) \to (a,b)} f(x,y) = L$ and $\lim_{(x,y) \to (a,b)} g(x,y) = M$.

Then, the following are true:
Computing Limits: Analytical Method

**Theorem**

1. **First, we have the obvious limits:**
   \[
   \lim_{(x,y) \to (a,b)} x = a, \quad \lim_{(x,y) \to (a,b)} y = b
   \]
   and
   \[
   \lim_{(x,y) \to (a,b)} c = c
   \]

2. **Sum and difference rules:**
   \[
   \lim_{(x,y) \to (a,b)} [f(x,y) \pm g(x,y)] = L \pm M
   \]

3. **Constant multiple rule:**
   \[
   \lim_{(x,y) \to (a,b)} [k f(x,y)] = kL
   \]

4. **Product rule:**
   \[
   \lim_{(x,y) \to (a,b)} [f(x,y) g(x,y)] = LM
   \]

5. **Quotient rule:**
   \[
   \lim_{(x,y) \to (a,b)} \left[ \frac{f(x,y)}{g(x,y)} \right] = \frac{L}{M} \text{ provided } M \neq 0.
   \]

6. **Power rule:** If \( r \) and \( s \) are integers with no common factors, and \( s \neq 0 \) then
   \[
   \lim_{(x,y) \to (a,b)} [f(x,y)]^{\frac{r}{s}} = L^{\frac{r}{s}} \text{ provided } L^{\frac{r}{s}} \text{ is a real number. If } s \text{ is even, we assume } L > 0.
   \]
Computing Limits: Analytical Method

Theorem

The above theorem applied to polynomials and rational functions implies the following:

1. To find the limit of a polynomial, we simply plug in the point.
2. To find the limit of a rational function, we plug in the point as long as the denominator is not 0.

Example

Find \( \lim_{(x,y) \to (1,2)} x^6 y + 2xy \)

Example

Find \( \lim_{(x,y) \to (1,1)} \frac{x^2 y}{x^4 + y^2} \)
Like for functions of one variable, the rules do not apply when plugging-in the point results in an indeterminate form. In that case, we must use techniques similar to the ones used for functions of one variable. Such techniques include factoring, multiplying by the conjugate. We illustrate them with examples.

**Example**

Find \( \lim_{(x,y) \to (0,0)} \frac{x^3 - y^3}{x - y} \)

**Example**

Find \( \lim_{(x,y) \to (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \)
The above examples correspond to cases where everything goes well. In other words, the limit exists and we can apply the limit rules to compute the limit. When the limit does not exist, one technique is to compute the limit along different paths and find two paths along which the limit has different values.

We look at several examples to see how this might be done. In general, you need to remember that specifying a path amounts to giving some relation between $x$ and $y$. When computing the limit along this path, use the relation which defines the path. For example, when computing the limit along the path $y = 0$, replace $y$ by 0 in the function. If computing the limit along the path $y = x$, replace $y$ by $x$ in the function. And so on...

Make sure that the path you select goes through the point at which we are computing the limit.
Limit Along a Path

**Example**
Compute \( \lim_{(x,y) \to (1,0)} \frac{y}{x+y-1} \) along the path \( x = 1 \).

**Example**
Compute \( \lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^2} \) along the path \( y = x^2 \).

**Example**
Prove that \( \lim_{(x,y) \to (1,0)} \frac{y}{x+y-1} \) does not exist.

**Example**
Prove that \( \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \) does not exist.
Limit Along a Path

Example
Prove that \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2} \) does not exist.

Example
Prove that \( \lim_{(x,y) \to (0,0)} \frac{x^2y}{x^4+y^2} \) does not exist.
Example

Using polar coordinates, find \( \lim_{{(x,y) \to (0,0)}} \frac{x^3+y^3}{x^2+y^2}. \)
We give two versions of the squeeze theorem and illustrate them with examples. The difficulty with the squeeze theorem is that we must suspect what the limit is. One way to know this is to compute the limit along various paths. As we saw earlier, if we find two paths along which the limit is different, we can conclude the limit does not exist. On the other hand, if we try several paths and we always get the same answer for the limit, we might suspect the limit exists. We then use the squeeze theorem to try to prove it.

**Theorem**

Suppose that \(|f(x, y) - L| \leq g(x, y)\) for every \((x, y)\) inside a disk centered at \((a, b)\), except maybe at \((a, b)\). If \(\lim_{(x,y)\to(a,b)} g(x, y) = 0\) then \(\lim_{(x,y)\to(a,b)} f(x, y) = L\).
### Additional Techniques: Squeeze Theorem

**Example**

Find \( \lim_{{(x,y) \to (0,0)}} f(x, y) \) for \( f(x, y) = \frac{x^2y}{x^2+y^2} \).

**Example**

Find \( \lim_{{(x,y) \to (1,0)}} f(x, y) \) for \( f(x, y) = \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} \).
Here is the second version of the squeeze theorem.

**Theorem**

If \( g(x, y) \leq f(x, y) \leq h(x, y) \) for all \((x, y) \neq (x_0, y_0)\) in a disk centered at \((x_0, y_0)\) and if \( \lim_{(x, y) \to (x_0, y_0)} g(x, y) = \lim_{(x, y) \to (x_0, y_0)} h(x, y) = L \) then \( \lim_{(x, y) \to (x_0, y_0)} f(x, y) = L \).

Here, the difficulty is to find the two functions \(g\) and \(h\) which satisfy the inequality and have a common limit. We illustrate this with an example.

**Example**

Does knowing that \( 2 |xy| - \frac{x^2y^2}{6} \leq 4 - 4 \cos \sqrt{|xy|} \leq 2 |xy| \) help you with finding \( \lim_{(x,y) \to (0,0)} \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|} \)?
Continuity

**Definition**

\( f(x, y) \) is **continuous** at a point \((a, b)\) if:

1. \((a, b)\) is in the domain of \(f\).
2. \(\lim_{(x,y) \to (a,b)} f(x, y)\) exists.
3. \(\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)\)

**Definition**

If a function \(f\) is not continuous at a point \((a, b)\), we say that it is **discontinuous** at \((a, b)\).

**Definition**

We say that a function \(f\) is continuous on a set \(D\) if it is continuous at every point in \(D\).
The following results are true for multivariable functions:

1. The sum, difference and product of continuous functions is a continuous function.
2. The quotient of two continuous functions is continuous as long as the denominator is not 0.
3. Polynomial functions are continuous.
4. Rational functions are continuous in their domain.
5. If \( f(x, y) \) is continuous and \( g(x) \) is defined and continuous on the range of \( f \), then \( g(f(x, y)) \) is also continuous.
### Example

Is \( f(x, y) = x^2 y + 3x^3 y^4 - x + 2y \) continuous at \((0, 0)\)? Where is it continuous?

### Example

Where is \( f(x, y) = \frac{2x-y}{x^2+y^2} \) continuous?

### Example

Where is \( f(x, y) = \frac{1}{x^2-y} \) continuous?

### Example

Find where \( \tan^{-1}\left(\frac{xy^2}{x+y}\right) \) is continuous.
Example

Find where \( \ln (x^2 + y^2 - 1) \) is continuous.

Example

Where is \( f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{at } (0, 0) \end{cases} \) continuous?

Example

Where is \( f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{at } (0, 0) \end{cases} \) continuous?
Exercises

See the problems at the end of my notes on limits and continuity of functions of two or more variables.
Review the notion of the derivative from Calculus I.