3.4 The Chain Rule

You will recall from Calculus I that we apply the chain rule when we have the composition of two functions, for example when computing \( \frac{d}{dx} \sin(e^x) \). The chain rule applies in similar situations when dealing with functions of several variables. For example, \( z = f(x, y) \) is a function of the two variables \( x \) and \( y \). However, \( x \) and \( y \) may be functions of one or more other variables. For example \( z = xy \) where \( x = \cos t \) and \( y = \sin t \). In this case, \( z \) is ultimately a function of \( t \).

3.4.1 Partials of \( f(x, y) \) where \( x \) and \( y \) are functions of one other variable \( t \)

Before we start, let us remind the reader that if a variable depends on several variables then the derivatives are partial derivatives and we will use the partial derivative notation as in \( \frac{\partial z}{\partial x} \) but if a variable depends only on one other variable, we will use the notation from calculus of functions of one variable as in \( \frac{dx}{dt} \).

Now, suppose we have \( z = f(x, y) \), \( x = g(t) \), \( y = h(t) \). Then, in this case we can also think of \( z \) as a function of \( t \). So, there is only one derivative to compute, \( \frac{dz}{dt} \). It can be computed as follows:

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \quad (3.1)
\]

To remember this formula, we can use a tree structure shown in figure 3.9 as follows.

1. Draw the tree from top to bottom. Assuming \( z = f(x, y) \), \( x = g(t) \), \( y = h(t) \), we start with \( z \).
2. From \( z \), we draw a branch for each variable \( z \) depends on, \( x \) and \( y \) in this case, so we draw two branches.
3. From each of these variables, we repeat the procedure, that is draw a branch for each variable it depends on. Both \( x \) and \( y \) only depend on \( t \), so we draw one branch from each.

The tree is interpreted as follows. Since \( z \) is ultimately a function of \( t \), look at all the paths from \( z \) to \( t \).

1. Each path is the partial derivative of the variable on top with respect to the variable at the bottom.
2. Multiply all the partials which appear on a path.
3. Add all these products collected on each path.
In our case, we obtain the following: There are two paths from \( z \) to \( t \). The first one is \( z \to x \to t \) which gives \( \frac{\partial z}{\partial x} \frac{dx}{dt} \). The second path is \( z \to y \to t \) which gives \( \frac{\partial z}{\partial y} \frac{dy}{dt} \). Adding these gives \( \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \) which is \( \frac{dz}{dt} \).

Similarly, if \( w = f (x, y, z) \) and \( x, y, z \) are functions of \( t \), then the corresponding tree structure is shown in figure 3.10.

Again, \( w \) is ultimately a function of \( t \). So, there is only one derivative to compute, \( \frac{dw}{dt} \). Using the interpretation outlines above, we obtain the following formula:

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}
\]

and so on.

**Example 3.4.1** Find \( \frac{du}{dt} \) if \( u = x^2 - y^2 \) and \( x = t^2 - 1, \ y = 3 \sin \pi t \).

\[
\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} \\
= \frac{\partial}{\partial x} (x^2 - y^2) \frac{dx}{dt} + \frac{\partial}{\partial y} (x^2 - y^2) \frac{dy}{dt} \\
= 2x \frac{dx}{dt} - 2y \frac{dy}{dt} \\
= 2(t^2 - 1) \frac{dt}{dt} - 2(3 \sin \pi t) \frac{\pi \cos \pi t}{dt} \\
= 2(t^2 - 1) - 6\pi y \cos \pi t \\
= 4(t^2 - 1) - 6\pi (3 \sin \pi t) \cos \pi t \\
= 4t^3 - 4t - 18\pi \sin \pi t \cos \pi t
\]
3.4. THE CHAIN RULE

3.4.2 Partials of \( f(x, y) \) where \( x \) and \( y \) are functions of two variables \( s \) and \( t \)

We can use the same tree structure as above to compute partial derivatives in this case. Suppose we have \( z = f(x, y), \) \( x = g(s, t), \) \( y = h(s, t) \). So, in this case, \( z \) is a function of \( s \) and \( t \). So, there are two partial derivatives to compute: \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \). The corresponding tree structure is shown in figure 3.11.

The first partials of \( z \) with respect to \( x \) or \( t \) are computed as follows:

\[
\begin{align*}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}
\end{align*}
\]

(3.2)  (3.3)

Example 3.4.2 Let \( z = \sin(x + y) \) where \( x = 2st \) and \( y = s^2 + t^2 \). Find \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \).

\[
\begin{align*}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\
&= \cos(x + y)(2t) + \cos(x + y)(2s) \\
&= 2t \cos(2st + s^2 + t^2) + 2s \cos(st + s^2 + t^2) \\
&= 2 \cos((s + t)^2)(s + t)
\end{align*}
\]
and

\[ \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \cos(x+y)(2s) + \cos(x+y)(2t) = 2 \cos\left((s+t)^2\right)(s+t) \]

**Example 3.4.3** Let \( u = x^2 - 2xy + 2y^3 \) where \( x = s^2 \ln t \) and \( y = 2st^3 \). Find \( \frac{\partial u}{\partial s} \) and \( \frac{\partial u}{\partial t} \)

\[ \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} = (2x-2y)(2s \ln t) + (-2x+6y^2)(2t^3) = (2s^2 \ln t - 4st^3)(2s \ln t) + (-2s^2 \ln t + 24s^2 t^6)(2t^3) \]

\[ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} = (2s^2 \ln t - 4st^3)\left(\frac{s^2}{t}\right) + (-2s^2 \ln t + 24s^2 t^6)(6st^2) \]

**Example 3.4.4** Given \( z = f(x,y) \), \( x = r^2 + s^2 \) and \( y = 2rs \) find \( \frac{\partial z}{\partial r} \) and \( \frac{\partial^2 z}{\partial r^2} \)
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- Computation of $\frac{\partial z}{\partial r}$.

\[
\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}
\]
\[
= 2r \frac{\partial z}{\partial x} + 2s \frac{\partial z}{\partial y}
\]

- Computation of $\frac{\partial^2 z}{\partial r^2}$

\[
\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial r} \right)
\]
\[
= \frac{\partial}{\partial r} \left( 2r \frac{\partial z}{\partial x} + 2s \frac{\partial z}{\partial y} \right)
\]
\[
= 2 \frac{\partial r}{\partial r} \left( \frac{\partial z}{\partial x} \right) + 2r \frac{\partial r}{\partial r} \left( \frac{\partial z}{\partial x} \right) + 2 \left( \frac{\partial s}{\partial r} \right) \left( \frac{\partial z}{\partial y} \right) + 2s \frac{\partial r}{\partial r} \left( \frac{\partial z}{\partial y} \right)
\]
\[
= 2 \frac{\partial z}{\partial x} + 2r \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial x} \right) + 2s \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial y} \right)
\]
\[
(3.4)
\]

We compute separately

\[
\frac{\partial}{\partial r} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) \frac{\partial y}{\partial r}
\]
\[
= 2r \frac{\partial^2 z}{\partial x^2} + 2s \frac{\partial^2 z}{\partial y \partial x}
\]
\[
(3.5)
\]

and

\[
\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) \frac{\partial y}{\partial r}
\]
\[
= 2r \frac{\partial^2 z}{\partial x \partial y} + 2s \frac{\partial^2 z}{\partial y^2}
\]
\[
(3.6)
\]

If $f$ has continuous second partial derivatives, then combining Equations 3.4, 3.5 and 3.6 gives

\[
\frac{\partial^2 z}{\partial r^2} = 2 \frac{\partial z}{\partial x} + 4r^2 \frac{\partial^2 z}{\partial x^2} + 8rs \frac{\partial^2 z}{\partial y \partial x} + 4s^2 \frac{\partial^2 z}{\partial y^2}
\]

If $f$ does not have continuous partial derivatives, then we cannot combine the mixed partials, and we get an expression slightly more complicated:

\[
\frac{\partial^2 z}{\partial r^2} = 2 \frac{\partial z}{\partial x} + 4r^2 \frac{\partial^2 z}{\partial x^2} + 4rs \frac{\partial^2 z}{\partial y \partial x} + 4s^2 \frac{\partial^2 z}{\partial x \partial y} + 4s^2 \frac{\partial^2 z}{\partial y^2}
\]
3.4.3 General Case

Suppose \( f \) is a function of \( n \) variables \( x_1, x_2, \ldots, x_n \) and each \( x_i \) is in turn a function of \( m \) variables \( t_1, t_2, \ldots, t_m \). \( f \) is then a function of \( m \) variables. The \( m \) first partials of \( f \) are:

\[
\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial t_i}
\]

for each \( i = 1, 2, \ldots, m \).

3.4.4 Implicit Differentiation

The chain rule can be used to derive a simpler method for finding the derivative of an implicitly defined function.

\( y \) defined implicitly as a function of \( x \) in a relation of the form \( F(x, y) = 0 \)

Suppose that \( F(x, y) = 0 \) defines \( y \) as an implicit function of \( x \) we will call \( y = f(x) \). We wish to find \( \frac{dy}{dx} \). We do so by differentiating both sides of \( F(x, y) = 0 \) with respect to \( x \). The right side is easy to differentiate and we will skip the details here! To differentiate the left side with respect to \( x \), \( F(x, y) \), we will use the chain rule, remembering that \( F(x, y) = F(x, f(x)) \). So, \( F \) is ultimately a function of \( x \). The corresponding tree structure is shown in figure \ref{fig:3.4.3}.

We obtain:
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Example 3.4.5 Find \( \frac{dy}{dx} \) if \( 2xy - y^3 + 1 - x - 2y = 0 \).

Using the notation above, \( F(x, y) = 2xy - y^3 + 1 - x - 2y \)

\[
\frac{dy}{dx} = \frac{-F_x}{F_y}
\]

\[
= \frac{-2y - 1}{2x - 3y^2 - 2}
\]

z defined implicitly as a function of x and y in a relation of the form \( F(x, y, z) = 0 \)

Suppose that \( F(x, y, z) = 0 \) defines z as an implicit function of x and y that is \( F(x, y, z) = F(x, y, f(x, y)) \). We can use a tree structure to find the partial derivatives. It is shown in figure 3.13. The dotted lines indicate that if x were a function of y, it is where we would have drawn a branch. Since x and y are the independent variables, they do not depend on each other. So, x is not a function of y and y is not a function of x.

If we differentiate each side with respect to x, using the tree structure, we
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get
\[ \frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial z} \frac{dz}{dx} = 0 \]

Now, \( \frac{\partial x}{\partial x} = 1 \) so we get
\[ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{dz}{dx} = 0 \]

That is
\[ \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \]

Similarly, we can show that
\[ \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z} \]

Example 3.4.6 Find \( \frac{\partial z}{\partial x} \) if \( z \) is defined implicitly by
\[ x^3 + y^3 + z^3 + 6xyz = 1 \]

Using the formula above with \( F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1 \), we see that
\[ \frac{\partial z}{\partial x} = \frac{F_x}{F_z} = \frac{3x^2 + 6yz}{3z^2 + 6xy} = \frac{x^2 + 2yz}{z^2 + 2xy} \]

Compare this solution with the solution of example 3.3.10 on page 213.

3.4.5 Problems

Make sure you have read, studied and understood what was done above before attempting the problems.

For the problems below, do the following:

• Find \( \frac{dw}{dt} \) using the chain rule.

• Find \( \frac{dw}{dt} \) by first expressing \( w \) as a function of \( t \).

• Evaluate \( \frac{dw}{dt} \) at the given value of \( t \).

1. \( w = x^2 + y^2 \), \( x = \cos t \), \( y = \sin t \), \( t = \pi \).
2. \( w = \frac{x}{z} + \frac{y}{z} \), \( x = \cos^2 t \), \( y = \sin^2 t \), \( z = \frac{1}{t} \), \( t = 3 \).

3. \( w = 2ye^x - \ln z \), \( x = \ln \left(t^2 + 1\right) \), \( y = \tan^{-1} t \), \( z = e^t \), \( t = 1 \).

For the problems below, do the following:

- Find \( \frac{\partial z}{\partial u} \) and \( \frac{\partial z}{\partial v} \) using the chain rule.
- Find \( \frac{\partial z}{\partial u} \) and \( \frac{\partial z}{\partial v} \) by first expressing \( z \) as a function of \( u \) and \( v \).
- Evaluate \( \frac{\partial z}{\partial u} \) and \( \frac{\partial z}{\partial v} \) at the given point \( (u, v) \).

4. \( z = 4e^x \ln y \), \( x = \ln (u \cos v) \), \( y = u \sin v \), \( (u, v) = \left(2, \frac{\pi}{4}\right) \).

For the problems below, do the following:

- Find \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) using the chain rule.
- Find \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) by first expressing \( w \) as a function of \( u \) and \( v \).
- Evaluate \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) at the given point \( (u, v) \).

5. \( w = xy + yz + xz \), \( x = u + v \), \( y = u - v \), \( z = uv \), \( (u, v) = \left(\frac{1}{2}, 1\right) \).

For the problems below, do the following:

- Find \( \frac{\partial u}{\partial x} \), \( \frac{\partial u}{\partial y} \) and \( \frac{\partial u}{\partial z} \) using the chain rule.
- Find \( \frac{\partial u}{\partial x} \), \( \frac{\partial u}{\partial y} \) and \( \frac{\partial u}{\partial z} \) by first expressing \( u \) as a function of \( u \) and \( v \).
- Evaluate \( \frac{\partial u}{\partial x} \), \( \frac{\partial u}{\partial y} \) and \( \frac{\partial u}{\partial z} \) at the given point \( (x, y, z) \).

6. \( u = \frac{p - q}{q - r} \), \( p = x + y + z \), \( q = x - y + z \), \( r = x + y - z \), \( (x, y, z) = \left(\sqrt{3}, 2, 1\right) \).

Draw a tree diagram and write a formula for each derivative below.

7. \( \frac{dz}{dt} \) for \( z = f(x, y) \), \( x = g(t) \), \( y = h(t) \).

8. \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) for \( w = h(x, y, z) \), \( x = f(u, v) \), \( y = g(u, v) \) and \( z = k(u, v) \).
9. \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) for \( w = g(x, y), x = g(u, v) \) and \( y = k(u, v) \).

10. \( \frac{\partial z}{\partial t} \) and \( \frac{\partial z}{\partial s} \) for \( z = f(x, y), x = g(t, s) \) and \( y = h(t, s) \).

11. \( \frac{\partial w}{\partial s} \) and \( \frac{\partial w}{\partial t} \) for \( w = g(u), u = h(s, t) \).

12. \( \frac{\partial w}{\partial r} \) and \( \frac{\partial w}{\partial s} \) for \( w = f(x, y), x = g(r) \) and \( y = h(s) \).

Assuming the relations below define \( y \) as an implicit function of \( x \), find \( \frac{dy}{dx} \) at the given point.

13. \( x^3 - 2y^2 + xy = 0 \), \((1, 1)\)

14. \( x^2 + xy + y^2 - 7 = 0 \), \((1, 2)\)

Assuming the relations below define \( z \) as an implicit function of \( x \) and \( y \), find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) at the given point.

15. \( z^3 - xy + yz + y^3 - 2 = 0 \), \((1, 1, 1)\).

16. \( \sin(x + y) + \sin(y + z) + \sin(x + z) = 0 \), \((\pi, \pi, \pi)\).

17. Find \( \frac{\partial w}{\partial r} \) when \( r = 1, s = -1 \), if \( w = (x + y + z)^2, x = r - s, y = \cos(r + s), z = \sin(r + s) \).

Additional problems.

18. If \( z = f(x, y) \) where \( f \) is differentiable, \( x = g(t), y = h(t), g(3) = 2, g'(3) = 5, h(3) = 7, h'(3) = -4, f_x(2, 7) = 6, f_y(2, 7) = -8 \), find \( \frac{dz}{dt} \) when \( t = 3 \).

19. The pressure of 1 mole of perfect gas is increasing at a rate of 0.05 kPa/s and the temperature is increasing at a rate of 0.15 K/s. Knowing that \( PV = 8.31T \) find \( \frac{dV}{dt} \) when \( P = 20kPa \) and \( T = 320K \).

20. If \( z = f(x, y) \) where \( x = r \cos \theta \) and \( y = r \sin \theta \)

1. Find \( \frac{\partial z}{\partial r} \) and \( \frac{\partial z}{\partial \theta} \).

2. Show that \( \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 = \left( \frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial z}{\partial \theta} \right)^2 \)

21. If \( z = f(x - y) \), show that \( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \) (hint: let \( u = x - y \) then use the chain rule to compute the partial derivatives).
### 3.4.6 Answers

For the problems below, do the following:

- Find $\frac{dw}{dt}$ using the chain rule.
- Find $\frac{dw}{dt}$ by first expressing $w$ as a function of $t$.
- Evaluate $\frac{dw}{dt}$ at the given value of $t$.

1. $w = x^2 + y^2$, $x = \cos t$, $y = \sin t$, $t = \pi$.
   
   - $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$
   
   - $w = \cos^2 t + \sin^2 t = 1$ hence $\frac{dw}{dt} = 0$.
   
   - $\frac{dw}{dt} \bigg|_{t=\pi} = 0$

2. $w = \frac{x}{z} + \frac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, $z = \frac{1}{t}$, $t = 3$.
   
   - $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$
   
   - $w = t$
   
   Hence
   
   - $\frac{dw}{dt} \bigg|_{t=3} = 1$

3. $w = 2ye^x - \ln z$, $x = \ln \left(t^2 + 1\right)$, $y = \tan^{-1} t$, $z = e^t$, $t = 1$.
   
   - $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$
   
   - $= 4t \tan^{-1} t + 1$
\[ w = 2 \tan^{-1} t \left( t^2 + 1 \right) - t \]

So
\[ \frac{dw}{dt} = 4t \tan^{-1} t + 1 \]

\[ \left. \frac{dw}{dt} \right|_{t=1} = 4 \tan^{-1} 1 + 1 = \pi + 1 \]

For the problems below, do the following:

- Find \( \frac{\partial z}{\partial u} \) and \( \frac{\partial z}{\partial v} \) using the chain rule.
- Find \( \frac{\partial z}{\partial u} \) and \( \frac{\partial z}{\partial v} \) by first expressing \( z \) as a function of \( u \) and \( v \).
- Evaluate \( \frac{\partial z}{\partial u} \) and \( \frac{\partial z}{\partial v} \) at the given point \((u, v)\).

4. \[ z = 4e^x \ln y, \quad x = \ln (u \cos v), \quad y = u \sin v, \quad (u, v) = \left(2, \frac{\pi}{4}\right). \]

\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = 4 \cos v \ln \left( u \sin v \right) + 1
\]

and
\[
\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = -4u \sin v \ln \left( u \sin v \right) + \frac{4u \cos^2 v}{\sin v}
\]

So
\[ \frac{\partial z}{\partial u} = 4 \cos v \ln \left( u \sin v \right) + 1 \]

and
\[ \frac{\partial z}{\partial v} = -4u \sin v \ln \left( u \sin v \right) + \frac{4u \cos^2 v}{\sin v} \]

\[ \left. \frac{\partial z}{\partial u} \right|_{\left(2, \frac{\pi}{4}\right)} = \sqrt{2} \left( \ln 2 + 2 \right) \]

\[ \left. \frac{\partial z}{\partial v} \right|_{\left(2, \frac{\pi}{4}\right)} = -2\sqrt{2} \left( \ln 2 - 2 \right) \]
3.4. THE CHAIN RULE

For the problems below, do the following:

- Find \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) using the chain rule.

- Find \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) by first expressing \( w \) as a function of \( u \) and \( v \).

- Evaluate \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) at the given point \((u, v)\).

5. \( w = xy + yz + xz, \) \( x = u + v, \) \( y = u - v, \) \( z = uv, \) \( (u, v) = \left(\frac{1}{2}, 1\right)\).

\[
\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} = 4uv + 2u
\]

and

\[
\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v} = -2v + 2u^2
\]

- \( w = u^2 - v^2 + 2u^2v \)

So

\[
\frac{\partial w}{\partial u} = 2u + 4uv
\]

and

\[
\frac{\partial w}{\partial v} = -2v + 2u^2
\]

- \( \frac{\partial w}{\partial u} \left(\frac{1}{2}, 1\right) = 3 \)

and

\[
\frac{\partial w}{\partial v} \left(\frac{1}{2}, 1\right) = \frac{-3}{2}
\]

For the problems below, do the following:

- Find \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \) and \( \frac{\partial u}{\partial z} \) using the chain rule.

- Find \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \) and \( \frac{\partial u}{\partial z} \) by first expressing \( u \) as a function of \( u \) and \( v \).
CHAPTER 3. FUNCTIONS OF SEVERAL VARIABLES

• Evaluate \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \) and \( \frac{\partial u}{\partial z} \) at the given point \((x, y, z)\).

6. \( u = \frac{p - q}{q - r}, \) \( p = x + y + z, \) \( q = x - y + z, \) \( r = x + y - z, \) \((x, y, z) = (\sqrt{3}, 2, 1)\).

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} = 0
\]

and

\[
\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} = \frac{z}{(z - y)^2}
\]

and

\[
\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial z} = \frac{-y}{(z - y)^2}
\]

\[
u = \frac{y}{z - y}
\]

So

\[
\frac{\partial u}{\partial x} = 0
\]

and

\[
\frac{\partial u}{\partial y} = \frac{z}{(z - y)^2}
\]

and

\[
\frac{\partial u}{\partial z} = \frac{-y}{(z - y)^2}
\]

•

\[
\frac{\partial u (\sqrt{3}, 2, 1)}{\partial x} = 0
\]

and

\[
\frac{\partial u (\sqrt{3}, 2, 1)}{\partial y} = 1
\]

and

\[
\frac{\partial u (\sqrt{3}, 2, 1)}{\partial z} = -2
\]
3.4. THE CHAIN RULE

Draw a tree diagram and write a formula for each derivative below.

7. \( \frac{dz}{dt} \) for \( z = f(x, y), x = g(t), y = h(t) \).

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

8. \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) for \( w = h(x, y, z), x = f(u, v), y = g(u, v) \) and \( z = k(u, v) \).

\[
\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}
\]

and

\[
\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}
\]

9. \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) for \( w = g(x, y), x = g(u, v) \) and \( y = k(u, v) \).

\[
\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u}
\]

and

\[
\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v}
\]

10. \( \frac{\partial z}{\partial t} \) and \( \frac{\partial z}{\partial s} \) for \( z = f(x, y), x = g(t, s) \) and \( y = h(t, s) \).

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}
\]

and

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}
\]

11. \( \frac{\partial w}{\partial s} \) and \( \frac{\partial w}{\partial t} \) for \( w = g(u), u = h(s, t) \).

\[
\frac{\partial w}{\partial s} = \frac{dw}{du} \frac{du}{ds}
\]

and

\[
\frac{\partial w}{\partial t} = \frac{dw}{du} \frac{du}{dt}
\]

12. \( \frac{\partial w}{\partial r} \) and \( \frac{\partial w}{\partial s} \) for \( w = f(x, y), x = g(r) \) and \( y = h(s) \).

\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r}
\]

and

\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}
\]
Assuming the relations below define $y$ as an implicit function of $x$, find $\frac{dy}{dx}$ at the given point.

13. $x^3 - 2y^2 + xy = 0$, $(1, 1)$

$$\frac{dy}{dx} = \frac{-3x^2 + y}{-4y + x}$$

Hence

$$\frac{dy (1, 1)}{dx} = \frac{4}{3}$$

14. $x^2 + xy + y^2 - 7 = 0$, $(1, 2)$

$$\frac{dy}{dx} = \frac{2x + y}{x + 2y}$$

Hence

$$\frac{dy (1, 2)}{dx} = \frac{-4}{-5}$$

Assuming the relations below define $z$ as an implicit function of $x$ and $y$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the given point.

15. $z^3 - xy + yz + y^3 - 2 = 0$, $(1, 1, 1)$.

$$\frac{\partial z}{\partial x} = \frac{-y}{3z^2 + y}$$

So

$$\frac{\partial z (1, 1, 1)}{\partial x} = \frac{1}{4}$$

and

$$\frac{\partial z}{\partial y} = \frac{-x + z + 3y^2}{3z^2 + y}$$

So

$$\frac{\partial z (1, 1, 1)}{\partial y} = \frac{-3}{4}$$

16. $\sin (x + y) + \sin (y + z) + \sin (x + z) = 0$, $(\pi, \pi, \pi)$.

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = -1$$

and

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = -1$$
17. Find \( \frac{\partial w}{\partial r} \) when \( r = 1, s = -1, \) if \( w = (x + y + z)^2, x = r - s, y = \cos(r + s), \)
\( z = \sin(r + s). \)

\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} \\
= 2(r - s + \cos(r + s) + \sin(r + s))(1 - \sin(r + s) + \cos(r + s))
\]

So
\[
\frac{\partial w (1, -1)}{\partial r} = 12
\]

Additional Problems

18. If \( z = f(x, y) \) where \( f \) is differentiable, \( x = g(t), y = h(t), g'(3) = 2, \) 
\( g'(3) = 5, h(3) = 7, h'(3) = -4, f_x (2, 7) = 6, f_y (2, 7) = -8, \) find \( \frac{dz}{dt} \)
when \( t = 3. \)

\[
\frac{dz}{dt} = \frac{\partial f (x, y)}{\partial x} \frac{dx}{dt} + \frac{\partial f (x, y)}{\partial y} \frac{dy}{dt} \\
\text{When } t = 3, \text{ we have}
\]
\[
\frac{dz}{dt} = (6)(5) + (-8)(-4) \\
= 62
\]

19. The pressure of 1 mole of perfect gas is increasing at a rate of 0.05 kPa/s 
and the temperature is increasing at a rate of 0.15 K/s. Knowing that 
\( PV = 8.31T \) find \( \frac{dV}{dt} \) when \( P = 20kPa \) and \( T = 320K. \)

We have \( V = \frac{8.31T}{P} \) so
\[
\frac{dV}{dt} = \frac{\partial V}{\partial T} \frac{dT}{dt} + \frac{\partial V}{\partial P} \frac{dP}{dt} \\
= -0.27008L/s
\]

20. If \( z = f(x, y) \) where \( x = r \cos \theta \) and \( y = r \sin \theta \)

1. (a) Find \( \frac{\partial z}{\partial r} \) and \( \frac{\partial z}{\partial \theta}. \)

\[
\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} \\
= \cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y}
\]

and
\[
\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} \\
= -r \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \frac{\partial z}{\partial y}
\]
(b) Show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2$

The right hand side is

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y}\right)^2 + \frac{1}{r^2} \left(-r \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \frac{\partial z}{\partial y}\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

21. If $z = f(x - y)$, show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ (hint: let $u = x - y$ then use the chain rule to compute the partial derivatives).

Let $u = x - y$, then $\frac{\partial z}{\partial x} = \frac{dz}{du} \frac{\partial u}{\partial x} = \frac{dz}{du} \frac{d}{dx}$ and $\frac{\partial z}{\partial y} = \frac{dz}{du} \frac{\partial u}{\partial y} = -\frac{dz}{du}$.

Therefore,

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{dz}{du} \frac{d}{dx} - \frac{dz}{du}

= 0$$

$$f_x = \cos (x + ct) - 2 \sin (2x + 2ct)$$

so

$$f_{xx} = -\sin (x + ct) - 4 \cos (2x + 2ct)$$

$$= -(\sin (x + ct) + 4 \cos (2x + 2ct))$$

Hence the result.
Bibliography

