3.6 Directional Derivatives and the Gradient Vector

3.6.1 Functions of two Variables

Directional Derivatives

Let us first quickly review, one more time, the notion of rate of change. Given \( y = f(x) \), the quantity

\[
\frac{f(x + h) - f(x)}{h} = \frac{f(x) - f(a)}{x - a}
\]

is the rate of change of \( f \) with respect to \( x \). It studies how \( f \) changes when \( x \) changes. When we take the limit of the above quantity as \( h \to 0 \) or as \( x \to a \), then we have the instantaneous rate of change which is also called the derivative.

Similarly, if \( y = f(x, y) \), then the quantity

\[
\frac{f(x + h, y) - f(x, y)}{h}
\]

studies how \( f \) changes with \( x \) or in the direction of \( x \). The quantity

\[
\frac{f(x, y + h) - f(x, y)}{h}
\]

studies how \( f \) changes with \( y \) or in the direction of \( y \). When we take the limit of the above quantities as \( h \to 0 \), we have the instantaneous rate of change of \( f \) with respect to \( x \) and \( y \) respectively. These instantaneous rates of change are also called the partials of \( f \) with respect to \( x \) and \( y \) respectively. They are denoted \( f_x \) or \( f_y \). These rates of change only study how \( f \) changes when either \( x \) or \( y \) is changing. Since \( f \) is a function of both \( x \) and \( y \), both \( x \) and \( y \) are likely to change at the same time. So, we also need to study how \( f \) changes with respect to both \( x \) and \( y \). In other words, we also need to study the rate of change of \( f \) in any direction, not just the direction of \( x \) or \( y \).

Let \( \overrightarrow{u} = (a, b) \) be a non-zero unit vector. We wish to study how \( f \) changes in the direction of \( \overrightarrow{u} \). If we start at \((x, y)\) and move \( h \) units in the direction of \( \overrightarrow{u} \) to a point \((x', y')\), then the rate of change is given by

\[
\frac{f(x', y') - f(x, y)}{h}
\]

We need to find what \( x' \) and \( y' \) are. They are easy to find. If \( x' = x + ah \) and \( y' = y + bh \) then we will have moved by

\[
\sqrt{a^2h^2 + b^2h^2} = h\sqrt{a^2 + b^2} = h
\]
since \( \mathbf{u} \) is a unit vector. So, we see that the rate of change of \( f \) in the direction of the unit vector \( \mathbf{u} \) is given by

\[
\frac{f(x + ah, y + bh) - f(x, y)}{h}
\]

If we take the limit as \( h \to 0 \), then we have the instantaneous rate of change of \( f \) in the direction of \( \mathbf{u} \). So, we have the following definition:

**Definition 265** The directional derivative of \( f \) at a point \((x_0, y_0)\) in the direction of the unit vector \( \mathbf{u} = (a, b) \) is given by:

\[
D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}
\]

assuming this limit exists.

**Definition 266** The directional derivative of \( f \) at any point \((x, y)\) in the direction of the unit vector \( \mathbf{u} = (a, b) \) is given by:

\[
D_{\mathbf{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + ah, y + bh) - f(x, y)}{h}
\]

assuming this limit exists.

**Example 267** Find the derivative of \( f(x, y) = x^2 + y^2 \) in the direction of \( \mathbf{u} = (1, 2) \) at the point \((1, 1)\).

First, since \( \mathbf{u} \) is not a unit vector, we must replace it with a unit vector in the same direction. Such a vector is

\[
\mathbf{u} = \frac{1}{\| \mathbf{u} \|} (1, 2)
\]

The directional derivative is

\[
\lim_{h \to 0} \frac{f\left(1 + \frac{1}{\sqrt{5}} h, 1 + \frac{2}{\sqrt{5}} h\right) - f(1, 1)}{h} = \lim_{h \to 0} \frac{\left(1 + \frac{1}{\sqrt{5}} h\right)^2 + \left(1 + \frac{2}{\sqrt{5}} h\right)^2 - 2}{h}
\]

\[
= \lim_{h \to 0} \frac{1 + \frac{2h}{\sqrt{5}} + h^2 + 1 + \frac{4h}{\sqrt{5}} + 4h^2 - 2}{h}
\]

\[
= \lim_{h \to 0} \frac{6h}{\sqrt{5} + h}
\]

\[
= \frac{6}{\sqrt{5}}
\]

**Remark 268** It is important to use a unit vector for the direction in which we want to compute the derivative.
It turns out that we do not have to compute a limit every time we need to compute a directional derivative. We have the following theorem:

**Theorem 269** If \( f \) is a differentiable function in \( x \) and \( y \), then \( f \) has a directional derivative in the direction of any unit vector \( \mathbf{u} = (a, b) \) and

\[
D_{\mathbf{u}} f(x, y) = f_x(x, y) a + f_y(x, y) b
\]  
(3.13)

If \( \mathbf{u} \) makes an angle \( \theta \) with the positive \( x \)-axis, then we also have

\[
D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta
\]  
(3.14)

**Proof.** We begin by proving the second part.

1. Proof of equation 3.14. Since \( \mathbf{u} \) is a unit vector, if it makes an angle \( \theta \) with the positive \( x \)-axis, we can write \( \mathbf{u} = (\cos \theta, \sin \theta) \). The result follows from equation 3.13.

2. Proof of equation 3.13. We prove that for an arbitrary point \( (x_0, y_0) \) we have:

\[
D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0) a + f_y(x_0, y_0) b
\]

Let us define the function \( g \) by

\[
g(h) = f (x_0 + ha, y_0 + hb)
\]

Then, we see that

\[
g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}
\]

\[
= \lim_{h \to 0} \frac{f (x_0 + ha, y_0 + hb) - f (x_0, y_0)}{h}
\]

\[
= D_{\mathbf{u}} f(x_0, y_0)
\]  
(3.15)

On the other hand, we can also write \( g(h) = f(x, y) \) where \( x = x_0 + ah \) and \( y = y_0 + bh \). \( f \) is a function of \( x \) and \( y \). But since both \( x \) and \( y \) are functions of \( h \), \( f \) is also a function of \( h \). Using the chain rule (see previous section), we have

\[
g'(h) = \frac{df}{dh}
\]

\[
= \frac{\partial f}{\partial x} \frac{dx}{dh} + \frac{\partial f}{\partial y} \frac{dy}{dh}
\]

\[
= f_x(x, y) a + f_y(x, y) b
\]

Therefore,

\[
g'(0) = f_x(x_0, y_0) a + f_y(x_0, y_0) b
\]  
(3.16)

From equations 3.15 and 3.16, we see that

\[
D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0) a + f_y(x_0, y_0) b
\]

Since this is true for any \( (x_0, y_0) \), the result follows.
Example 270  Find the derivative of \( f(x, y) = x^2 + y^2 \) in the direction of \( \vec{u} = (1, 2) \) at the point \( (1, 1, 2) \).

This is the example we did above, using limits. Like above, we must use the unit vector having the same direction. Such a vector is

\[
\vec{u} = \frac{1}{\sqrt{5}} (1, 2)
\]

Therefore, \( D_{\vec{u}} f (1, 1) = f_x (1, 1) \frac{1}{\sqrt{5}} + f_y (1, 1) \frac{2}{\sqrt{5}} \). First, we compute the partials of \( f \)

\[
f_x (x, y) = 2x
\]

Therefore,

\[
f_x (1, 1) = 2
\]

Similarly,

\[
f_y (1, 1) = 2
\]

Therefore,

\[
D_{\vec{u}} f (1, 1) = \frac{2}{\sqrt{5}} + \frac{4}{\sqrt{5}} = \frac{6}{\sqrt{5}}
\]

The Gradient Vector

The above formula, \( f_x (x, y) a + f_y (x, y) b \) can be written as \( \langle f_x (x, y), f_y (x, y) \rangle \cdot \langle a, b \rangle \). The vector on the left has a special name: the gradient vector.

Definition 271  If \( f \) is a function of two variables in \( x \) and \( y \), then the gradient of \( f \), denoted \( \nabla f \) (read "grad \( f \)" or "del \( f \)"") is defined by:

\[
\nabla f (x, y) = \langle f_x (x, y), f_y (x, y) \rangle
\]

Example 272  Compute the gradient of \( f(x, y) = \sin x e^y \).

\[
\nabla f (x, y) = \langle f_x (x, y), f_y (x, y) \rangle = \langle \cos xe^y, \sin xe^y \rangle
\]

Example 273  Compute \( \nabla f (0, 1) \) for \( f(x, y) = x^2 + y^2 + 2xy \).

\[
\nabla f (x, y) = \langle f_x (x, y), f_y (x, y) \rangle = \langle 2x + 2y, 2y + 2x \rangle
\]

Therefore,

\[
\nabla f (0, 1) = \langle 2, 2 \rangle
\]
We can express the directional derivative in terms of the gradient.

**Theorem 274** If $f$ is a differentiable function in $x$ and $y$, then $f$ has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$ D_\mathbf{u} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} \tag{3.17} $$

**Proof.** We know that

$$ D_\mathbf{u} f(x_0, y_0) = f_x(x_0, y_0) a + f_y(x_0, y_0) b = \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle = \nabla f(x, y) \cdot \mathbf{u} $$


**Remark 275** From formula 3.17, we can recover the formulas for the partials of $f$ with respect to $x$ and $y$. For example, the partial of $f$ with respect to $x$ is the directional derivative of $f$ in the direction of $\mathbf{i} = (1, 0)$. So, we have

$$ D_\mathbf{i} f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle \cdot (1, 0) = f_x(x, y) $$

So, we can use formula 3.17 to compute the derivative in any direction, including $x$ and $y$.

### 3.6.2 Functions of three Variables

What we have derived above also applies to functions of three variables. Given a function $f(x, y, z)$ and a vector $\mathbf{u} = \langle a, b, c \rangle$, we have the following:

- $D_\mathbf{u} f(x, y, z) = \lim_{h \to 0} \frac{f(x+ah, y+bh, z+ch) - f(x, y, z)}{h}$.

- If we write $\mathbf{x} = (x, y, z)$, then we can write $D_\mathbf{u} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x}+h \mathbf{u}) - f(\mathbf{x})}{h}$ and this works for functions of 2 or 3 variables.

- $\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$.

- We can write $D_\mathbf{u} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$.

**Example 276** Find the directional derivative of $f(x, y, z) = x \cos y \sin z$ at $(1, \pi, \frac{\pi}{4})$ in the direction of $\mathbf{u} = \langle 2, -1, 4 \rangle$.

First, we find a unit vector having the same direction. Such a vector is $\frac{1}{\sqrt{21}} \langle 2, -1, 4 \rangle$.

Next, we compute

$$ \nabla f = \langle \cos y \sin z, -x \sin y \sin z, x \cos y \cos z \rangle $$
So,

\[ \nabla f \left( 1, \pi, \frac{\pi}{4} \right) = \left\langle -\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2} \right\rangle \]

It follows that the directional derivative we seek is:

\[
D_{\vec{u}} f \left( 1, \pi, \frac{\pi}{4} \right) = \nabla f \left( 1, \pi, \frac{\pi}{4} \right) \cdot \frac{1}{\sqrt{21}} \left( 2, -1, 4 \right) \\
= \left\langle -\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2} \right\rangle \cdot \frac{1}{\sqrt{21}} \left( 2, -1, 4 \right) \\
= -\frac{3\sqrt{2}}{\sqrt{21}} \\
\approx -0.92582
\]

### 3.6.3 Maximizing the directional derivative

As we saw above, the gradient can be used to find the directional derivative. It has many more applications. One such application is that we can use the gradient to find the direction in which a function has the largest rate of change. If you think of the graph of a function as a 2-D surface in 3-D, or a terrain on which you are walking, then the gradient can be used to find the direction in which the terrain is the steepest. Of course, depending on what you are trying to achieve, this may be the direction you want to avoid!! This can be accomplished as follows.

**Theorem 277** Suppose that \( f \) is a differentiable function and \( \vec{u} \) is a unit vector. The maximum value of \( D_{\vec{u}} f \) at a given point is \( \| \nabla f \| \) and it occurs when \( \vec{u} \) has the same direction as \( \nabla f \) at the given point.

**Proof.** We have already proven that

\[
D_{\vec{u}} f = \nabla f \cdot \vec{u} \\
= \| \nabla f \| \| \vec{u} \| \cos \theta
\]

where \( \theta \) is the smallest angle between \( \vec{u} \) and \( \nabla f \). Since \( \vec{u} \) is a unit vector, we have

\[
D_{\vec{u}} f = \| \nabla f \| \cos \theta
\]

So, we see that this is maximum when \( \cos \theta \) is maximum, that is when \( \cos \theta = 1 \).

In this case, we have

\[
D_{\vec{u}} f = \| \nabla f \|
\]

Since it happens when \( \cos \theta = 1 \), it happens when \( \theta = 0 \) that is when \( \vec{u} \) has the same direction as \( \nabla f \). \( \blacksquare \)

**Remark 278** Using a similar argument, we see that the minimum value of \( D_{\vec{u}} f \) at a given point is \(- \| \nabla f \| \) and it occurs when \( \vec{u} \) has the direction of \(- \nabla f \) at the given point.
3.6. DIRECTIONAL DERIVATIVES AND THE GRADIENT VECTOR

Example 279 Suppose that the temperature at each point of a metal plate is given by

\[ T(x, y) = e^x \cos y + e^y \cos x \]

1. In what direction does the temperature increase most rapidly at the point (0,0)? What is the rate on increase?

2. In what direction does the temperature decrease most rapidly at the point (0,0)?

- Solution to #1:
  \[ \nabla f(x, y) = (e^x \cos y - e^y \sin x, e^y \cos x - e^x \sin y) \]
  At (0,0) the temperature increases most rapidly in the direction of
  \[ \nabla f(0,0) = (1, 1) \]
  The rate of increase is
  \[ \|\nabla f(0,0)\| = \sqrt{2} \]

- Solution to #2: At (0,0), temperature decreases most rapidly in the direction of
  \[ -\nabla f(0,0) = (-1, -1) \]
  A picture of the gradient field around the origin is shown in figure 3.13. We plotted the gradient at each point near the origin. Remember that the gradient is a vector, it is represented as an arrow. The length of each arrow is the magnitude of the gradient and its direction, the direction of the gradient.

3.6.4 The Gradient as a Normal: Tangent Planes and Normal Lines to a Level Surface

The Gradient Vector and Level Curves.

Given a function of two variables \( z = f(x, y) \), its graph is a surface in \( \mathbb{R}^3 \). If \( k \) is any constant, then the graph of \( f(x, y) = k \) is a curve. It is called the level curves of \( f(x, y) \). Geometrically, it is the intersection of the graph of \( z = f(x, y) \) with the plane \( z = k \).

Example 280 The graph of \( z = f(x, y) = x^2 + y^2 \) is a paraboloid. The intersection of a paraboloid with a plane \( z = k \) where \( k > 0 \) is the circle centered at the origin of radius \( \sqrt{k} \). It is easy to see. If \( z = k \) and \( z = x^2 + y^2 \) then \( x^2 + y^2 = k \). This is the equation of the circle centered at the origin of radius \( \sqrt{k} \). Figure 3.14 shows the graph of \( z = f(x, y) = x^2 + y^2 \) as well as the level curves \( f(x, y) = k \) for \( k = 1, k = 2, k = 3, k = 4 \) and \( k = 5 \). Since these level curves are 2-D objects which live in a plane parallel to the xy-plane, the figure also shows their projection in the xy-plane.
The gradient of $f$ is $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$. It is a 2-D vector. It has the remarkable property that it is orthogonal to the level curves $f(x,y) = k$. This is summarized in the following proposition.

**Proposition 281** Let $z = f(x,y)$ be a function whose partial derivatives in $x$ and $y$ exist. Let $k$ be any constant. Let $C$ denote the level curves $f(x,y) = k$. Let $P = (x_0, y_0)$ be a point on $C$. Then $\nabla f(x_0, y_0)$ is orthogonal to $C$ at $P$.

**Proof.** To show this, we show that $\nabla f(x_0, y_0)$ is orthogonal to the tangent vector to $C$ at $P$. Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ be the position vector of the curve $C$. Let $t_0$ be the value of $t$ such that $\mathbf{r}(t_0) = \langle x_0, y_0 \rangle$. Then,

$$f(x(t), y(t)) = k$$

Thus

$$\frac{df(x,y)}{dt} = 0$$

Using the chain rule, we have

$$\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 0$$

that is

$$\nabla f(x,y) \cdot \mathbf{r}'(t) = 0$$

In particular, at $(x_0, y_0)$ we have

$$\nabla f(x_0, y_0) \cdot \mathbf{r}'(t_0) = 0$$
Figure 3.14: Graph of $z = x^2 + y^2$ and level curves
Example 282 Looking back at the previous example, \( f(x, y) = x^2 + y^2 \) and considering the level curve \( x^2 + y^2 = 4 \). It is a circle of radius 2 centered at the origin. Without computations, we know that a vector perpendicular to this circle at the point \((2, 0)\), that is at the intersection of the curve and the x-axis is a vector parallel to \( \mathbf{i} \). We can verify it using our theorem. The theorem says that such a vector would be \( \nabla f (2, 0) \). \( \nabla f (x, y) = \langle 2x, 2y \rangle \) hence \( \nabla f (2, 0) = \langle 2, 0 \rangle = 2 \mathbf{i} \), so it is parallel to \( \mathbf{i} \). Using the theorem, we also see that \( \nabla f (\sqrt{2}, \sqrt{2}) = \langle 2\sqrt{2}, 2\sqrt{2} \rangle = 2\sqrt{2} \mathbf{i} \) this means that a vector perpendicular to our curve at the point \( (\sqrt{2}, \sqrt{2}) \) is parallel to \( (1, 1) \). We could have predicted that because \( (\sqrt{2}, \sqrt{2}) \) is the point of intersection between our curve and the line \( y = x \). A direction vector for this line will be perpendicular to the curve. Such a vector is \( (1, 1) \). Figure 3.15 shows the surface \( z = x^2 + y^2 \), its level curves, and the two gradient vectors computed above, one is in red, the other one in green. One can see the gradient vectors are indeed perpendicular to the level curves.

The Gradient Vector and Level Surfaces

There is a similar result for level surfaces. Given a function of three variables \( F(x, y, z) \), the graph of \( F(x, y, z) = k \) is a surface. It is called the level surface of the function \( F(x, y, z) \). Suppose that \( S \) is a level surface of a function \( F(x, y, z) \) with equation \( F(x, y, z) = k \) where \( k \) is a constant. Let \( P = (x_0, y_0, z_0) \) be a point on \( S \). We have the following proposition:
Proposition 283 $\nabla F (x_0, y_0, z_0)$ is orthogonal to $S$ at $P$.

**Proof.** We prove the result by proving that $\nabla F (x_0, y_0, z_0)$ is orthogonal to any curve on $S$ through $P$. Let $C$ be any curve on $S$ through $P$ given by its position vector $\mathbf{r} (t) = (x(t), y(t), z(t))$. Let $t_0$ be the value of $t$ such that $\mathbf{r} (t_0) = (x_0, y_0, z_0)$, the coordinates of $P$ in other words $t_0$ is the value of the parameter for which the curve is at $P$. Because $C$ is on $S$, we have

$$F (x(t), y(t), z(t)) = k$$

Since $x, y, z$ are differentiable of $t$, $F$ is also a differentiable function of $t$. Using the chain rule, and differentiating both sides with respect to $t$, we have

$$\frac{dF}{dt} = 0$$

$$\frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0$$

Since $\nabla F = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$ and $\mathbf{r}' (t) = \left( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$, the above equation can be written as

$$\nabla F (x, y, z) \cdot \mathbf{r}' (t) = 0$$

In particular, at $(x_0, y_0, z_0)$ we have

$$\nabla F (x_0, y_0, z_0) \cdot \mathbf{r}' (t_0) = 0$$

Thus, $\nabla F (x_0, y_0, z_0)$ is orthogonal to the tangent vector of any curve through $P$. Since these tangent vectors are on the tangent plane, it follows that $\nabla F (x_0, y_0, z_0)$ is orthogonal to $S$ at $P$. 

**Example 284** Find a vector perpendicular to the surface $4x^2 + 2y^2 + z^2 = 16$ at the point $(1, 2, 2)$.

We define $F (x, y, z) = 4x^2 + 2y^2 + z^2$ and the surface can be thought of a level surface to this function, more specifically, the level surface corresponding to $F (x, y, z) = 16$. A vector perpendicular to this surface is $\nabla f (1, 2, 2) = (8x, 4y, 2z)$ hence $\nabla f (1, 2, 2) = (8, 8, 4)$.

**Tangent Plane to a Level Surface**

The technique developed here is not to be confused with work done in previous sections. Earlier, we learned how to find the tangent plane to a surface given by $z = f (x, y)$ at the point $(x_0, y_0, z_0)$. You will recall that the equation of such a plane is

$$z - z_0 = f_x (x_0, y_0) (x - x_0) + f_y (x_0, y_0) (y - y_0) \quad (3.18)$$

In this subsection, we learn how to find the equation of the tangent plane to a level surface $S$ of a function $F (x, y, z)$ at a point $P = (x_0, y_0, z_0)$, that is a surface given by $F (x, y, z) = k$ where $k$ is a constant. This plane is defined by $P$ and a vector perpendicular to $S$ at $P$. The problem is how to find such a
vector. In the past, when we have needed a vector perpendicular to a plane, if we knew two vectors on the plane, we took their cross product. In this case, to find a vector perpendicular to \( S \) at \( P \) or to the tangent plane to \( S \) at \( P \), we could apply the same idea, that is find two non parallel vectors on the tangent plane. Their cross product would give us the vector normal we are seeking. Unfortunately, we do not have such vectors. We do not have three non colinear points to generate them either. However, from the previous subsection, we know how to find a vector perpendicular to \( S \) at \( P \). Such a vector is \( \nabla F(x_0, y_0, z_0) \).

Now that we have a point on the tangent plane: \((x_0, y_0, z_0)\) and a normal vector \( \nabla F(x_0, y_0, z_0) \), it follows that the equation of the plane tangent to \( S \) at \( P \) is:

\[
F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0
\]

(3.19)

**Remark 285** We can use equation 3.19 to derive equation 3.18, the equation of the tangent plane to a surface given by \( z = f(x, y) \). If we rewrite \( z = f(x, y) \) as \( f(x, y) - z = 0 \), then we can think of the graph of \( f(x, y) \) as a level surface of the function \( F(x, y, z) = 0 \) where \( F(x, y, z) = f(x, y) - z \). In this case, \( F_x = f_x, F_y = f_y \) and \( F_z = -1 \). Thus, using equation 3.19, we see that the tangent plane is

\[
f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) - (z - z_0) = 0
\]

which is the same as equation 3.18.

**Example 286** Find an equation for the tangent plane to the elliptic cone \( x^2 + 4y^2 = z^2 \) at the point \((3, 2, 5)\).

We can rewrite the equation of the cone as \( x^2 + 4y^2 - z^2 = 0 \). Thus, the elliptic cone is a level surface of \( F(x, y, z) = x^2 + 4y^2 - z^2 \), it is the level surface corresponding to \( F(x, y, z) = 0 \). Since \( F_x = 2x, F_y = 8y \) and \( F_z = -2z \), from equation 3.19, it follows that the equation of the tangent plane is

\[
F_x(3, 2, 5)(x - 3) + F_y(3, 2, 5)(y - 2) + F_z(3, 2, 5)(z - 5) = 0
\]

\[
6(x - 3) + 16(y - 2) - 10(z - 5) = 0
\]

\[
6x - 18 + 16y - 32 - 10z + 50 = 0
\]

\[
6x + 16y - 10z = 0
\]

\[
3x + 8y - 5z = 0
\]

We illustrate this by graphing the level surface \( x^2 + 4y^2 - z^2 = 0 \), and its tangent plane at \((3, 2, 5)\) which is: \( 3x + 8y - 5z = 0 \). This is shown in figure 3.16.

**Normal line to a Level Surface**

Since \( \nabla F(x_0, y_0, z_0) \) is orthogonal to \( S \) at \( P \), it is the direction vector of the line normal to \( S \) at \( P \). The equation of such a line is

\[
\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}
\]

(3.20)
Example 287 Find the parametric equations for the normal line to the elliptic cone \( x^2 + 4y^2 = z^2 \) at the point \((3, 2, 5)\).

We can rewrite the equation of the cone as \( x^2 + 4y^2 - z^2 = 0 \). Thus, the elliptic cone is a level surface of \( F(x, y, z) = x^2 + 4y^2 - z^2 \), it is the level surface corresponding to \( F(x, y, z) = 0 \). Since \( F_x = 2x \), \( F_y = 8y \) and \( F_z = -2z \), from equation 3.20 and the previous exercise, it follows that the equation of the normal line is:

\[
\frac{x - 3}{6} = \frac{y - 2}{16} = \frac{z - 5}{-10}
\]

Multiplying each side by 2 gives

\[
\frac{x - 3}{3} = \frac{y - 2}{8} = \frac{z - 5}{-5}
\]

These are symmetric equations. The parametric equations for this line are

\[
\begin{align*}
x &= 3 + 3t \\
y &= 2 + 8t \\
z &= 5 - 5t
\end{align*}
\]

Summary About the Gradient Vector

We have studied the following about the gradient vector:
• \( \nabla f = \langle f_x, f_y \rangle \) for functions of two variables and \( \nabla f = \langle f_x, f_y, f_z \rangle \) for functions of 3 variables.

• The derivative of \( f \) in the direction of the **unit** vector \( \vec{u} \) is \( D_{\vec{u}} f(x) = \nabla f(x) \cdot \vec{u} \).

• The maximum value of \( D_{\vec{u}} f(x) \) is \( \| \nabla f(x) \| \), it happens in the direction of \( \nabla f(x) \). In other words, \( \nabla f(x) \) gives the direction of fastest increase for \( f \).

• \( \nabla f(x_0, y_0) \) is orthogonal to the level curves \( f(x, y) = k \) that passes through \( P = (x_0, y_0) \).

• \( \nabla F(x_0, y_0, z_0) \) is orthogonal to the tangent vector of any curve in \( S \) through \( P \) where \( S \) is the level surface \( F(x, y, z) = k \) and \( P = (x_0, y_0, z_0) \).

• \( \nabla F(x_0, y_0, z_0) \) is orthogonal to \( S \) at \( P \).

• The equation of the tangent plane to \( S \) at \( P \) is \( F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \).

• \( \nabla F(x_0, y_0, z_0) \) is the direction vector of the line normal to \( S \) at \( P \).

• The equation of the normal line to \( S \) at \( P \) is \( \frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)} \).

• If \( f \) and \( g \) are functions of 2 or 3 variables and \( c \) is a constant, it can be shown that the gradient satisfies the following properties (what do these properties remind you of?):

1. \( \nabla (f + g) = \nabla f + \nabla g \)
2. \( \nabla (f - g) = \nabla f - \nabla g \)
3. \( \nabla (cf) = c\nabla f \)
4. \( \nabla (fg) = f\nabla g + g\nabla f \)
5. \( \nabla \left( \frac{1}{g} \right) = \frac{g\nabla f - f\nabla g}{g^2} \)
6. \( \nabla f^n = nf^{n-1}\nabla f \)

### 3.6.5 Assignment

Do odd # 1 - 35 at the end of 11.5 in your book.

Do odd # 1 - 7 at the end of 11.6 in your book.