4.3 Double Integrals Over General Regions

4.3.1 Introduction

When dealing with integrals of functions of one variable, we are always integrating over an interval. The only difficulty in evaluating the definite integral \( \int_a^b f(x) \, dx \) came from the function \( f \) and the difficulty to find an antiderivative for it. With functions of two or more variables, not only the function can cause the integral to be difficult, but also the region over which we integrate. In the previous section, we restricted ourselves to rectangular regions. However, not every region is rectangular. A region of \( \mathbb{R}^2 \) can have any shape. Even if the function is easy to integrate, if the region is complex enough, the integral will still be difficult to evaluate. We first define the integral of a function over two variables over a general region. We will then show how this integral can be evaluated over some simple regions which we will call regions of types I and II.

Let us assume we are given a function \( f(x,y) \) defined on a closed and bounded region \( D \) as shown in figure 4.7. Suppose we want to integrate \( f \) over \( D \). Because \( D \) is bounded, it can be enclosed in a rectangle \( R \) as shown in figure 4.8.

To define the integral of \( f \) over \( D \), we first define the function \( F(x,y) \) as follows:

\[
F(x,y) = \begin{cases} 
  f(x,y) & \text{if } (x,y) \in D \\
  0 & \text{if } (x,y) \in R - D
\end{cases}
\]  

(4.3)
Definition 4.3.1 If \( \iint_R F(x,y) \, dA \) exists, then we define

\[
\iint_D f(x,y) \, dA = \iint_R F(x,y) \, dA
\]

This definition makes sense since, as we can see in figures [4.9] and [4.10], the portion of the graph of \( F \) which is not 0 is identical to the graph of \( f \). The portion which is 0 will not contribute to the integral. In particular, this means that for our definition, it does not matter which rectangle \( R \) we select. In the case, \( f(x,y) \geq 0 \), \( \iint_D f(x,y) \, dA \) corresponds to the volume of the solid which lies above \( D \) and below the graph of \( z = f(x,y) \).

We still must be able to compute \( \iint_R F(x,y) \, dA \). This is not always a simple task. But it is for certain regions, which we consider next.

### 4.3.2 Regions of Type I

When describing a region, one has to give the condition \( x \) and \( y \) must satisfy so that a point \( (x,y) \) lies in the region. A region is said to be of type I if \( x \) is between two constants, and \( y \) is between two continuous functions of \( x \). More precisely, we have the following definition:

**Definition 4.3.2** A plane region \( D \) is said to be of **type I** if it is of the form

\[
D = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(x)\}
\]
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Figure 4.9: Graph of $f$

Figure 4.10: Graph of $F$
where \( g_1 \) and \( g_2 \) are two continuous functions of \( x \).

Possible regions of type I are shown in figures 4.11, 4.12 and 4.13. To evaluate \( \iint_D f(x,y) \, dA \), we enclose \( D \) in the rectangle \([a, b] \times [c, d]\) as shown in figure 4.3.2.

Imagine covering the region \( D \) with vertical strips of width \( dx \). To cover the whole region, we must use strips for \( a \leq x \leq b \). For each \( x \) (for each strip), we have \( g_1(x) \leq y \leq g_2(x) \).

We define \( F(x,y) \) as explained in equation 4.3. So, we have

\[
\iint_D f(x,y) \, dA = \int_R F(x,y) \, dA
= \int_a^b \int_c^d F(x,y) \, dy \, dx \quad \text{by Fubini's theorem}
\]

To evaluate this iterated integral, we first evaluate the inside integral. Using the properties of the definite integral, we have:

\[
\int_c^d F(x,y) \, dy = \int_c^{g_1(x)} F(x,y) \, dy + \int_{g_1(x)}^{g_2(x)} F(x,y) \, dy + \int_{g_2(x)}^d F(x,y) \, dy
= \int_{g_1(x)}^{g_2(x)} F(x,y) \, dy
\]

Figure 4.11: Region of type I
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Figure 4.12: Region of type I

Figure 4.13: Region of type I
since \( F(x, y) = 0 \) for \( y < g_1(x) \) and \( y > g_2(x) \) (see figure 4.3.2). Also when \( y \) is between \( g_1(x) \) and \( g_2(x) \), \( F(x, y) = f(x, y) \), so we have

\[
\int_c^d F(x, y) \, dy = \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy
\]

Therefore, we have the following proposition:

**Proposition 4.3.3** If \( f \) is continuous on a type I region \( D \) such that

\[
D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(x)\}
\]

then

\[
\iint_D f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx \tag{4.4}
\]

**Remark 4.3.4** Notice that the function \( F(x, y) \) does not appear in the formula. It was simply a device used to derive this formula. When computing an integral, it does not come into play, as the next example will show.

**Remark 4.3.5** When writing \( \iint_D f(x, y) \, dA \) as an iterated integral, it is important to remember that the outer integral must be the one with constant limits of integration.

**Remark 4.3.6** To evaluate a double integral over a general region, the first step is always to write it as an iterated integral. How this is done depends on the region. The region will determine what the limits of integration of the iterated integrals are. Hence, it is extremely important to really understand what the region looks like. Graphing it will help.
Example 4.3.7 Write $\int\int_D f(x, y) \, dA$ as an iterated integral where $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ and } -x^2 \leq y \leq x^2\}$.

Here, the region is given explicitly, we know the limits for both $x$ and $y$, so we can write the iterated integral.

$$\int\int_D f(x, y) \, dA = \int_{-x^2}^{x^2} \int_0^1 f(x, y) \, dy \, dx$$

Example 4.3.8 Write $\int\int_D f(x, y) \, dA$ as an iterated integral where $D$ is the region bounded by the $x$-axis, $y = \sin x$, the lines $x = 0$ and $x = 1$.

The region is shown in figure 4.3.8.

\begin{center}
\includegraphics[width=\textwidth]{Region_D}
\end{center}

We see that we can write $D$ as: $D = \{(x, y) \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq \sin x\}$ and hence

$$\int\int_D f(x, y) \, dA = \int_0^1 \int_0^{\sin x} f(x, y) \, dy \, dx$$

Example 4.3.9 Evaluate $\int\int_D (x^4 - 2y) \, dA$ where $D = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1 \text{ and } -x^2 \leq y \leq x^2\}$. 


$D$ is clearly a type I region.

\[
\iiint_D (x^4 - 2y) \, dA = \int_{-1}^{1} \int_{-x^2}^{x^2} (x^4 - 2y) \, dy \, dx
\]

\[
= \int_{-1}^{1} \left[ x^4y - y^2 \right]_{-x^2}^{x^2} \, dx
\]

\[
= \int_{-1}^{1} (x^6 - x^4 + x^6 + x^4) \, dx
\]

\[
= \int_{-1}^{1} 2x^6 \, dx
\]

\[
= \left[ \frac{2}{7} x^7 \right]_{-1}^{1}
\]

\[
= \frac{4}{7}
\]

### 4.3.3 Regions of Type II

The condition a point $(x, y)$ must satisfy to be in a type II region is as follows. A region is said to be of type II if $y$ is between two constants, and $x$ is between two continuous functions of $y$. More precisely, we have the following definition:

**Definition 4.3.10** A plane region $D$ is said to be of **type II** if it is of the form

\[
D = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d \text{ and } h_1(y) \leq x \leq h_2(y)\}
\]

Using the same method as for type I regions, we can show that

**Proposition 4.3.11** If $f$ is continuous on a type II region $D$ such that

\[
D = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d \text{ and } h_1(y) \leq x \leq h_2(y)\}
\]

then

\[
\iiint_D f(x, y) \, dA = \int_c^{d} \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy \tag{4.5}
\]

**Remark 4.3.12** This time, we imagine covering the region $D$ with horizontal strips of height $dy$. To cover the whole region, we must use strips for $c \leq y \leq d$. For each $y$ (for each strip), we have $h_1(y) \leq x \leq h_2(y)$.

**Example 4.3.13** Evaluate \( \iiint_D (xy - y^3) \, dA \) if $D$ is the region bounded by the $x$-axis, the lines $x = -1$, $y = 1$ and $y = x$. It often helps to sketch the region so we can see what type of region it is. This region is shown in figure [4.14]. This is a region of type II, it can be described
Figure 4.14: Region bounded by the $x$–axis, and the lines $y = -1$, $x = -1$, and $y = x$.

by $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 \text{ and } -1 \leq x \leq y\}$. In type II regions, $x$ is the variable between two functions of $y$. Make sure these are functions of $y$, not $x$. In this case, the right side of the region is delimited by the line $y = x$, which when we write as a function of $y$ becomes $x = y$. Therefore, we have

\[
\iint_D (xy - y^3) \, dA = \int_0^1 \int_{-1}^y (xy - y^3) \, dx \, dy
\]

\[
= \int_0^1 \left[ \frac{x^2}{2} - xy^3 \right]_{-1}^y \, dy
\]

\[
= \int_0^1 \left( \frac{y^3}{2} - y^4 - \left( \frac{y}{2} + y^3 \right) \right) \, dy
\]

\[
= \int_0^1 \left( -y^4 - \frac{y^3}{2} - \frac{y}{2} \right) \, dy
\]

\[
= \left( -\frac{y^5}{5} - \frac{y^4}{8} - \frac{y^2}{4} \right) \bigg|_0^1
\]

\[
= -\frac{23}{40}
\]

4.3.4 Properties

We list without proof several properties the double integral satisfies. These properties are very similar to the properties of the definite integral of functions of one variable.

**Proposition 4.3.14** Assuming that the following integrals exist and that $C$ is a constant, we have:
1. $\int\int_D (f(x, y) \pm g(x, y)) \, dA = \int\int_D f(x, y) \, dA \pm \int\int_D g(x, y) \, dA$

2. $\int\int_D C f(x, y) \, dA = C \int\int_D f(x, y) \, dA$

3. If $f(x, y) \geq g(x, y)$ for all $(x, y)$ in $D$, then $\int\int_D f(x, y) \, dA \geq \int\int_D g(x, y) \, dA$

**Proposition 4.3.15** If $D, D_1$ and $D_2$ are three regions in $\mathbb{R}^2$ such that $D = D_1 \cup D_2$ and $D_1$ and $D_2$ do not overlap then

$$\int\int_D f(x, y) \, dA = \int\int_{D_1} f(x, y) \, dA + \int\int_{D_2} f(x, y) \, dA$$

This property corresponds to the property for functions of one variable which says: $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$. It can sometimes be used to evaluate integrals over regions which are neither of type I, nor of type II. If we have such a region and we can divide it into two (or more regions) of type I or II, then we can break the integral into integrals over the subregions. Each integral is then an integral of type I or II, which we know how to evaluate.

**Proposition 4.3.16** $\int\int_D 1 \, dA = A(D)$ where $A(D)$ denotes the area of $D$.

This property says that if we integrate the constant function $f(x, y) = 1$ over a region $D$, we get the area of $D$. This makes sense. The volume of a solid a section $D$ between two parallel planes is (area of the section) $\times$ height. In this case, the height is simply 1. This is also very important as it gives us a way of computing the area of general regions. We illustrate it with an example.

**Example 4.3.17** Find the area of the region $D$ in the first quadrant bounded by $y = x$ and $y = x^2$.

The region is shown in figure 4.15. We can treat this as a type 1 or a type 2 region. We show the solution both ways.

- **Type 1 region.** The area of the region is $\int\int_D dA$ where $D$ is the region described in the problem, in green on figure 4.15. To evaluate this integral, we must rewrite it as an iterated integral treating $D$ as a type 1 region. We imagine covering the region with vertical strips as shown in figure 4.16.
Figure 4.15: Region between $y = x$ and $y = x^2$
We would use a strip for every $x$ such that $0 \leq x \leq 1$ and for each $x$, the range of $y$ values along the strip would be: $x^2 \leq y \leq x$. Therefore

\[
\int \int_D dA = \int_0^1 \int_{x^2}^x dy \, dx
\]

\[
= \int_0^1 y_{x^2}^x \, dx
\]

\[
= \int_0^1 (x - x^2) \, dx
\]

\[
= \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1
\]

\[
= \frac{1}{2} - \frac{1}{3}
\]

\[
= \frac{1}{6}
\]

- Type 2 region. This is similar to above, but we imagine covering the region with horizontal strips as shown in figure 4.17; we would use such strips
for $0 \leq y \leq 1$. For each $y$, the range of $x$ values along the strip would be $y \leq x \leq \sqrt{y}$. Therefore

$$\iint_D dA = \int_0^1 \int_y^{\sqrt{y}} dx \, dy$$

$$= \int_0^1 x\sqrt{y} \, dy$$

$$= \int_0^1 (\sqrt{y} - y) \, dy$$

$$= \left( \frac{2}{3}y^2 - \frac{y^3}{2} \right) \bigg|_0^1$$

$$= \frac{2}{3} - \frac{1}{2}$$

$$= \frac{1}{6}$$

Combining the above properties, we have:
Proposition 4.3.18 If \( m \leq f(x,y) \leq M \) for all \((x,y)\) in \(D\), then

\[ mA(D) \leq \int\int_D f(x,y) \, dA \leq MA(D) \]

Definition 4.3.19 (Average Value) We define the average value of a function \( f(x,y) \) over a region \(D\) to be

\[
\text{Average value of } f \text{ over } D = \frac{1}{\text{area of } D} \int\int_D f(x,y) \, dA
\]

4.3.5 More Practice Problems

In many cases, a region can be considered of type I or II. One then has to decide whether to treat it as a type I region or a type II. It may be that either way will work. However, in some cases, one way may lead to an integral which is very difficult, while the other way leads to an easier integral. In the example that follows, we look at cases to illustrate these various options. The more problem you practice on, the better you will become at identifying the type of a region, and how to perform the integration.

Example 4.3.20 Evaluate

\[
\int\int_{\Omega} (\sqrt{x} - y^2) \, dA
\]

where \(\Omega\) is the region bounded by \(y = x^2\) and \(y = x^{1/4}\).

Once again, it helps to draw the region. It is shown in figure 4.18.
To completely define the region, one need to find the points where the two regions meet. For this, we solve

\[
\begin{align*}
\begin{cases}
y = x^2 \\
y = \sqrt{x}
\end{cases}
\end{align*}
\]

Combining the two gives

\[x^2 = x^{\frac{1}{4}}\]

If we raise both sides to the fourth power, we get

\[
\begin{align*}
x^8 &= x \\
x^8 - x &= 0 \\
x(x^7 - 1) &= 0 \\
x &= 0 \text{ or } x = 1
\end{align*}
\]

When \(x = 0\), we get \(y = 0\). When \(x = 1\), we get \(y = 1\). We see that this region can be seen as a type I region: \(\Omega = \{(x, y) \mid 0 \leq x \leq 1 \text{ and } x^2 \leq y \leq x^{\frac{1}{4}}\}\). Note that \(y\) is between two functions of \(x\). \(\Omega\) can also be seen as a type II region. (see figure 4.19). In this case, \(\Omega = \{(x, y) \mid 0 \leq y \leq 1 \text{ and } y^4 \leq x \leq \sqrt{y}\}\). Note that \(x\) is between two functions of \(y\). We obtained them as follows. From figure 4.19, we see that the region is bounded on the left by \(y = x^{\frac{1}{4}}\) which can be written as \(x = y^4\). It is bounded on the right by \(y = x^2\) which can be written as \(x = \sqrt{y}\). We compute the integral both ways.
Method 1 Treating $\Omega$ as a type I region.

\[
\int_\Omega (\sqrt{x} - y^2) \, dA = \int_0^1 \int_{x^2}^{x^4} (\sqrt{x} - y^2) \, dy \, dx \\
= \int_0^1 \left[ y\sqrt{x} - \frac{y^3}{3} \right]_{x^2}^{x^4} \, dx \\
= \int_0^1 \left( x^2 - \frac{x^6}{3} \right) - \left( x^2 - \frac{x^6}{3} \right) \, dx \\
= \int_0^1 \left( \frac{2}{3} x^4 - x^6 + \frac{x^6}{3} \right) \, dx \\
= \left( \frac{8}{21} x^7 \frac{2}{7} + \frac{1}{21} \right) \bigg|_0 \\
= \frac{8}{21} \frac{2}{7} + \frac{1}{21} \\
= \frac{1}{7}
\]

Method 2 Treating $\Omega$ as a type II region.

\[
\int_\Omega (\sqrt{x} - y^2) \, dA = \int_0^1 \int_{y^4}^{y^2} (\sqrt{x} - y^2) \, dx \, dy \\
= \int_0^1 \left[ \left( \frac{2}{3} x^2 - xy^2 \right) \right]_{y^4}^{y^2} \, dy \\
= \int_0^1 \left( \frac{2}{3} y^2 - y^2 - \left( \frac{2}{3} y^6 - y^6 \right) \right) \, dy \\
= \int_0^1 \left( \frac{2}{3} y^2 - y^2 + \frac{y^6}{3} \right) \, dy \\
= \left( \left( \frac{8}{21} y^7 \frac{2}{7} + \frac{1}{21} \right) \right) \bigg|_0 \\
= \left( \frac{8}{21} \frac{2}{7} + \frac{1}{21} \right) \\
= \frac{1}{7}
\]

Example 4.3.21 Evaluate $\int_\Omega \cos \frac{\pi x^2}{2} \, dA$ where $D$ is the region bounded by $x = 1$, $y = 0$ and $y = x$. 

The region is shown in figure 4.20. Once again, we see that this region can be treated as a type I or type II region.

**Method 1** Treating $\Omega$ as a type I region. The corner points of the region are $(0,0), (1,0)$ and $(1,1)$. The region is determined by $\Omega = \{(x,y) | 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x\}$. In this case:

$$\iint_{\Omega} \cos \frac{\pi x^2}{2} \, dA = \int_0^1 \int_0^x \cos \frac{\pi x^2}{2} \, dy \, dx$$

$$= \int_0^1 \left[ y \cos \frac{\pi x^2}{2} \right]_0^x \, dx$$

$$= \int_0^1 x \cos \frac{\pi x^2}{2} \, dx$$

We finish with a substitution. If we let $u = \frac{\pi x^2}{2}$ then $du = \pi x \, dx$. Also, when $x = 0$, $u = 0$ and when $x = 1$, $u = \frac{\pi}{2}$. So, we have

$$\iint_{\Omega} \cos \frac{\pi x^2}{2} \, dA = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \cos u \, du$$

$$= \frac{1}{\pi} (\sin u)|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}$$

**Method 2** Treating $\Omega$ as a type II region. The region is determined by $\Omega =
\[ \{(x,y) \mid 0 \leq y \leq 1 \text{ and } y \leq x \leq 1\}. \] So

\[
\iint_{\Omega} \cos \frac{\pi x^2}{2} \, dA = \int_0^1 \int_y^1 \cos \frac{\pi x^2}{2} \, dx \, dy
\]

We cannot evaluate the inside integral as we do not know an antiderivative with respect to \( x \) of \( \int_y^1 \cos \frac{\pi x^2}{2} \, dx \).

**Example 4.3.22** Calculate the volume of the solid within the cylinder \( x^2 + y^2 = b^2 \), between the planes \( y + z = a \) and \( z = 0 \), given that \( a \geq b > 0 \).

The region is shown in figure 4.3.22. The region \( D \) in \( \mathbb{R}^2 \) by which the solid is bounded is \( D = \{(x,y) \mid x^2 + y^2 \leq b^2\} \). The solid is also bounded by the planes \( z = 0 \) and \( z = -y + a \). Therefore, its volume is given by the integral

\[
\iiint_D (a - y) \, dA = \iint_D adA - \iint_D ydA
\]

The second integral is easier to evaluate than it seems. The region \( D \) can be written as

\[
D = \{(x,y) \mid -b \leq x \leq b \text{ and } -\sqrt{b^2 - x^2} \leq y \leq \sqrt{b^2 - x^2}\}
\]

Thus,

\[
\iint_D ydA = \int_{-b}^{b} \int_{-\sqrt{b^2 - x^2}}^{\sqrt{b^2 - x^2}} y \, dy \, dx
\]

You will recall that integrating an odd function over a symmetric integral gives 0, in other words, if \( f(x) \) is an odd function, then \( \int_{-a}^{a} f(x) \, dx = 0 \). Thus,
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\[
\int_{\sqrt{x^2-x}}^{\sqrt{y^2-y}} y \, dy = 0. \quad \text{It follows that}
\]

\[
\int_D (a - y) \, dA = \int_D a \, dA = a \int_D dA = aA(D) \quad \text{by one of the properties above}
\]

\[
= \pi ab^2
\]

4.3.6 Switching the Order of Integration

Evaluating a double integral over a region amounts to evaluating one or more iterated integrals. As we know, there are two kinds of iterated integrals. One in which the \(dx\) integral is the inner integral and the \(dy\) integral is the outer integral. The other kind is the reverse. When the region of integration is a rectangle, Fubini’s theorem tells us that the two integrals are the same. If \(R = [a, b] \times [c, d]\). Then

\[
\int \int_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.
\]

When the region of integration is not a rectangle, we may not have a choice. We may be forced to use one of the iterated integrals as suggests example 4.3.21. That example also suggests that given an iterated integral, to evaluate it might require switching the order of integration. This cannot be done by just reversing the order of the integrals. It involves changing the type of region we are integrating over. We illustrate this with an example.

**Example 4.3.23** Switch the order of integration \(\int_0^1 \int_y^1 \cos \frac{\pi x^2}{2} \, dx \, dy\).

First, we need to identify the region, decide what type of region it is. Then, we rewrite it as a region of the other type and write the integral in terms of that region. From the limits of integration, we see that the region is \(D = \{(x, y) : 0 \leq y \leq 1 \text{ and } y \leq x \leq 1\}\). Thus, this is a type II region. Thus

\[
\int_0^1 \int_y^1 \cos \frac{\pi x^2}{2} \, dx \, dy = \int_D \cos \frac{\pi x^2}{2} \, dA \quad \text{where } D \text{ is as indicated and is shown in figure 4.20}.
\]

To rewrite as a type I region, we need to express \(x\) between two constants and \(y\) between two functions. From figure 4.20, we see that \(D\) can also be written as \(D = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x\}\). In that region,

\[
\int_D \cos \frac{\pi x^2}{2} \, dA = \int_0^1 \int_0^x \cos \frac{\pi x^2}{2} \, dy \, dx.
\]

Therefore,

\[
\int_0^1 \int_y^1 \cos \frac{\pi x^2}{2} \, dx \, dy = \int_D \cos \frac{\pi x^2}{2} \, dA = \int_0^1 \int_0^x \cos \frac{\pi x^2}{2} \, dy \, dx.
\]

Hence

\[
\int_0^1 \int_y^1 \cos \frac{\pi x^2}{2} \, dx \, dy = \int_0^1 \int_0^x \cos \frac{\pi x^2}{2} \, dy \, dx
\]
4.3.7 Summary

We list important points covered in this section.

- Since not every region in the plane is a rectangle, we generalized what we had learned in the previous section to more general regions, closed and bounded regions. If we call $D$ such a region, then we learned to compute

$$\int \int_D f(x,y) \, dA.$$ 

- It is very important to understand that for multiple integrals, the region of integration, $D$, plays a very important role. I cannot emphasize enough the importance of making sure one knows the region of integration well before attempting to compute $\int \int_D f(x,y) \, dA$. Every multiple integral should start with drawing $D$, the region of integration, and writing a concise mathematical description of it.

- We actually consider two types of regions. Type I regions and type II regions. Unlike rectangular regions, the order of integration cannot be switched easily, at least not without some work.

- Type I regions: $D = \{(x,y) \mid a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(x)\}$. In this case

$$\int \int_D f(x,y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx$$

Note that the limits of integration of the outer integral are always constants.

- Type II regions: $D = \{(x,y) \mid c \leq y \leq d \text{ and } h_1(y) \leq x \leq h_2(y)\}$. In this case

$$\int \int_D f(x,y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) \, dx \, dy$$

Note that the limits of integration of the outer integral are always constants.

- Given $\int \int_D f(x,y) \, dA$, students must be able to determine if $D$ is a type I or II region and write the double integral as the corresponding iterated integral.
• Given an iterated integral, that is one of \[ \int_{a}^{b} \int_{g_{1}(x)}^{g_{2}(x)} f(x, y) \, dy \, dx \] or \[ \int_{c}^{d} \int_{h_{1}(x)}^{h_{2}(x)} f(x, y) \, dx \, dy, \]
students must be able to find the region \( D \) such that the iterated integral is the same as \( \iint_{D} f(x, y) \, dA \).

• When \( D \), the region of integration, can be written as both a type I and a type II region, it is possible to switch the order of integration. Sometimes it is necessary to do so. To reverse the order of integration of an iterated integral, perform the following steps:
  – Given an iterated integral, find the region \( D \) that it corresponds to and identify the type of region \( D \) is.
  – Write \( D \) as a region of the other type.
  – Write the iterated integral corresponding to the newly written region.

• If the graph of \( z = f(x, y) \) is above the xy-plane then \( \iint_{D} f(x, y) \, dA \) is the volume of the solid with cross section \( D \), between the xy-plane and \( z = f(x, y) \).

• \( \iint_{D} dA \) gives the area of the region \( D \).

### 4.3.8 Problems

1. Sketch the region of integration then evaluate \( \int_{0}^{\pi} \int_{0}^{x} x \sin y \, dy \, dx \).

2. Sketch the region of integration then evaluate \( \int_{1}^{\infty} \int_{0}^{\ln y} e^{x+y} \, dx \, dy \).

3. Sketch the region of integration then evaluate \( \int_{0}^{1} \int_{0}^{y^2} 3y^3 e^{-y} \, dx \, dy \).

4. Find \( \iint_{D} \frac{x}{y} \, dA \) where \( D \) is the region in the first quadrant bounded by the lines \( y = x, y = 2x, x = 1, x = 2 \).

5. Find \( \iint_{D} (u - \sqrt{u}) \, dA \) where \( D \) is the triangular region cut from the first quadrant of the uv-plane by the line \( u + v = 1 \).

6. Sketch the region of integration then evaluate \( \int_{-2}^{0} \int_{v}^{v} 2 \, dp \, dv \) in the pv-plane.
7. Sketch the region of integration then evaluate \( \int_0^\frac{\pi}{2} \int_0^{\cos t} 3 \cos t \, dt \, du \) in the \( tu \)-plane.

8. Sketch the region of integration, then reverse the order of integration for \( \int_0^1 \int_2^{1-2x} dy \, dx \).

9. Sketch the region of integration, then reverse the order of integration for \( \int_0^1 \int_y^{\sqrt{\pi}} dx \, dy \).

10. Sketch the region of integration, then reverse the order of integration for \( \int_0^1 \int_0^t dy \, dx \).

11. Sketch the region of integration, then reverse the order of integration for \( \int_0^3 \int_0^{3-4x^2} 16xy \, dx \, dy \).

12. Sketch the region of integration, then reverse the order of integration for \( \int_0^1 \int_{\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y \, dx \, dy \).

13. Sketch the region of integration, then reverse the order of integration and evaluate \( \int_0^\pi \int_x^\pi \sin \frac{y}{y} \, dy \, dx \).

14. Sketch the region of integration, then reverse the order of integration and evaluate \( \int_0^1 \int_y^1 e^{xy^2} \, dx \, dy \).

15. Sketch the region of integration, then reverse the order of integration and evaluate \( \int_0^{2\sqrt{3}} \int_{2\sqrt{3}}^{2\sqrt{3}} e^{x^2} \, dx \, dy \).

16. Find the volume of the region bounded by the paraboloid \( z = x^2 + y^2 \) and below by the triangle enclosed by the lines \( y = x \), \( x = 0 \), and \( x + y = 2 \).

17. Find the volume of the solid whose base is the region in the \( xy \)-plane bounded by the parabola \( y = 4 - x^2 \) and the line \( y = 3x \) and bounded above by \( z = x + 4 \).

18. Sketch and find the area of the region bounded by the coordinate axes and \( x + y = 2 \).

19. Sketch and find the area of the region bounded by the parabola \( x = -y^2 \) and \( y = x + 2 \).

20. Sketch and find the area of the region bounded by \( y = e^x \), \( y = 0 \), \( x = 0 \), \( x = \ln 2 \).

21. Find the region and its area corresponding to \( \int_0^6 \int_{2y}^{x} \, dx \, dy \).

22. Find the region and its area corresponding to \( \int_0^{\frac{2\pi}{3}} \int_{\sin x}^{\cos x} \, dy \, dx \).
4.3. DOUBLE INTEGRALS OVER GENERAL REGIONS

4.3.9 Answers

1. The region is \( D = \{(x, y) : 0 \leq x \leq \pi \text{ and } 0 \leq y \leq x\} \). Hence \( \int_0^\pi \int_0^x x \sin y \, dy \, dx = \frac{\pi^2 + 4}{2} \).

2. The region is \( D = \{(x, y) : 1 \leq y \leq \ln 8 \text{ and } 0 \leq x \leq \ln y\} \). Hence \( \int_\ln 8^1 \int_0^{\ln y} e^{x+y} \, dx \, dy = 24 \ln 2 + e - 16 \).

3. The region is \( D = \{(x, y) : 0 \leq y \leq 1 \text{ and } 0 \leq x \leq y^2\} \). Hence \( \int_0^1 \int_0^{y^2} 3y^3 e^{xy} \, dx \, dy = e - 2 \).

4. \( \int \int_D \frac{x}{y} \, dA = \frac{3}{2} \ln 2 \).

5. \( \int \int_D (u - \sqrt{u}) \, dA = -\frac{1}{10} \).

6. \( \int_0^1 \int_v^1 2p \, dp \, dv = 8 \).

7. \( \int_0^\pi \int_0^{\sec t} 3 \cos t \, dt \, dv = 2\pi \).

8. \( \int_0^1 \int_2^{1-2x} dy \, dx = \int_0^4 \int_0^{2-x} dx \, dy \).

9. \( \int_0^1 \int_y^\pi dxdy = \int_0^1 \int_x^\pi dy \, dx \).

10. \( \int_0^1 \int_1^e dxdy = \int_0^1 \int_y^1 dx \, dy \).

11. \( \int_0^2 \int_0^{9-4x^2} 16x \, dy \, dx = \int_0^9 \int_0^{\frac{\sqrt{x-1}}{2}} 16x \, dx \, dy \).

12. \( \int_0^1 \int_0^{\sqrt{1-3y}} 3y \, dx \, dy = \int_0^1 \int_0^{1-\sqrt{3x}} 3y \, dx \, dy \).

13. \( \int_0^\pi \int_x^\sin y \frac{y}{y} \, dy \, dx = \int_0^\pi \int_0^{\sin y} \frac{y}{y} \, dx \, dy = 2 \).

14. \( \int_0^1 \int_y^1 x^2 e^{xy} \, dx \, dy = \int_0^1 \int_0^x x^2 e^{xy} \, dy \, dx = \frac{e - 2}{2} \).

15. \( \int_0^{2\sqrt[3]{3}} \int_0^{\sqrt{3x^2}} e^x \, dx \, dy = \int_0^{\sqrt[3]{3}} \int_0^{2x} e^x \, dx \, dy = 2 \).

16. \( V = \int_0^1 \int_x^{-x+2} (x^2 + y^2) \, dy \, dx = \frac{4}{3} \).

17. \( V = \int_1^4 \int_{3x}^{4-x^2} (x + 4) \, dy \, dx = \frac{625}{12} \).

18. \( A = \int_0^2 \int_0^{-x+2} \, dy \, dx = 2 \).
19. \[ A = \int_{-2}^{1} \int_{y-2}^{-y^2} dx \, dy = \frac{9}{2}. \]

20. \[ A = \int_{0}^{\ln 2} \int_{0}^{e^x} dy \, dx = 1. \]

21. The region is \( D = \{(x, y) : 0 \leq y \leq 6 \text{ and } \frac{y^2}{3} \leq x \leq 2y\} \). Its area is \( A = \int_{0}^{6} \int_{\frac{y^2}{3}}^{2y} dx \, dy = 12. \)

22. The region is \( D = \{(x, y) : \sin x \leq y \leq \cos x \text{ and } 0 \leq x \leq \frac{\pi}{4}\} \). Its area is \( A = \int_{0}^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} dy \, dx = \sqrt{2} - 1. \)