2.4 Tangent, Normal and Binormal Vectors

Three vectors play an important role when studying the motion of an object along a space curve. These vectors are the unit tangent vector, the principal normal vector and the binormal vector. We have already defined the unit tangent vector. In this section, we define the other two vectors.

Let us start by reviewing the definition of the unit tangent vector.

**Definition 147 (Unit Tangent Vector)** Let \(C\) be a smooth curve with position vector \(\mathbf{r}(t)\). The **unit tangent vector**, denoted \(\mathbf{T}(t)\), is defined to be

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}
\]

### 2.4.1 Normal and Binormal Vectors

**Definition 148 (Normal Vector)** Let \(C\) be a smooth curve with position vector \(\mathbf{r}(t)\). The **principal normal vector** or simply the **normal vector**, denoted \(\mathbf{N}(t)\), is defined to be:

\[
\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}
\]  \hspace{1cm} (2.13)

The name of this vector suggests that it is normal to something, the question is to what? By definition, \(\mathbf{T}\) is a unit vector, that is \(\|\mathbf{T}(t)\| = 1\). From proposition 120, it follows that \(\mathbf{T}'(t) \perp \mathbf{T}(t)\). Thus, \(\mathbf{N}(t) \perp \mathbf{T}(t)\). In fact, \(\mathbf{N}(t)\) is a unit vector, perpendicular to \(\mathbf{T}\) pointing in the direction where the curve is bending.

**Definition 149 (Binormal Vector)** Let \(C\) be a smooth curve with position vector \(\mathbf{r}(t)\). The **binormal vector**, denoted \(\mathbf{B}(t)\), is defined to be

\[
\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)
\]

Since both \(\mathbf{T}(t)\) and \(\mathbf{N}(t)\) are unit vectors and perpendicular, it follows that \(\mathbf{B}(t)\) is also a unit vector. It is perpendicular to both \(\mathbf{T}(t)\) and \(\mathbf{N}(t)\).

**Example 150** Consider the circular helix \(\mathbf{r}(t) = (\cos t, \sin t, t)\). Find the unit tangent, normal and binormal vectors.

- **Unit Tangent:** Since \(\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\), we need to compute \(\mathbf{r}'(t)\) and \(\|\mathbf{r}'(t)\|\).

  \[
  \mathbf{r}'(t) = (-\sin t, \cos t, 1)
  \]
and

\[ \vec{r}'(t) = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} \]

Thus

\[ \vec{T}(t) = \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \]

- Normal: Since \( \vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \), we need to compute \( \vec{T}'(t) \) and \( \|\vec{T}'(t)\| \).

\[ \vec{T}'(t) = \left\langle \frac{-\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, 0 \right\rangle \]

and

\[ \|\vec{T}'(t)\| = \sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2}} = \frac{1}{\sqrt{2}} \]

Thus

\[ \vec{N}(t) = (-\cos t, -\sin t, 0) \]

- Binormal:

\[ \vec{B}(t) = \vec{T}(t) \times \vec{N}(t) \]

\[ = \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \times \left\langle -\cos t, -\sin t, 0 \right\rangle \]

\[ \vec{B}(t) = \left\langle \frac{\sin t - \cos t}{\sqrt{2}}, \frac{\cos t - \sin t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \]

The pictures below (figures 2.5, 2.6 and 2.7) show the helix for \( t \in [0, 2\pi] \) as well as the three vectors \( \vec{T}(t) \), \( \vec{N}(t) \) and \( \vec{B}(t) \) plotted for various values of \( t \). If the three vectors do not appear to be exactly orthogonal, it is because the scale is not the same in the \( x, y \) and \( z \) directions.

2.4.2 Osculating and Normal Planes

**Definition 151 (Osculating and Normal Planes)** Let \( C \) be a smooth curve with position vector \( \vec{r}(t) \). Let \( P \) be a point on the curve corresponding to \( \vec{r}(t_0) \) for some value of \( t \).

1. The plane through \( P \) determined by \( \vec{N}(t_0) \) and \( \vec{B}(t_0) \) is called the normal plane of \( C \) at \( P \). Note that its normal will be \( \vec{T}(t) \).
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Figure 2.5: Helix and the vectors $\vec{T}(0)$, $\vec{N}(0)$ and $\vec{B}(0)$

Figure 2.6: Helix and the vectors $\vec{T}(1)$, $\vec{N}(1)$ and $\vec{B}(1)$
2. The plane through $P$ determined by $\vec{T}(t_0)$ and $\vec{N}(t_0)$ is called the osculating plane of $C$ at $P$. Note that its normal will be $\vec{B}(t)$.

**Example 152** Find the normal and osculating planes to the helix given by $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ at the point $\left(0, 1, \frac{\pi}{2}\right)$.

Earlier, we found that

$$\vec{T}(t) = \left\langle -\sin t, \cos t, \frac{1}{\sqrt{2}} \right\rangle$$

$$\vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$$

and

$$\vec{B}(t) = \left\langle \sin t, -\cos t, \frac{1}{\sqrt{2}} \right\rangle$$

At the point $\left(0, 1, \frac{\pi}{2}\right)$, that is when $t = \frac{\pi}{2}$, we have

$$\vec{T}\left(\frac{\pi}{2}\right) = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

$$\vec{N}\left(\frac{\pi}{2}\right) = \langle 0, -1, 0 \rangle$$
and
\[ \vec{B} \left( \frac{\pi}{2} \right) = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle \]

- **Normal Plane:** It is the plane through \( \left( 0, 1, \frac{\pi}{2} \right) \) with normal \( \vec{T} \left( \frac{\pi}{2} \right) = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle \). Thus its equation is
\[ -\frac{1}{\sqrt{2}} (x - 0) + \frac{1}{\sqrt{2}} \left( z - \frac{\pi}{2} \right) = 0 \]

Multiplying each side by \( \sqrt{2} \) gives
\[ -(x - 0) + \left( z - \frac{\pi}{2} \right) = 0 \]

or
\[ z - x = \frac{\pi}{2} \]

- **Osculating Plane:** It is the plane through \( \left( 0, 1, \frac{\pi}{2} \right) \) with normal \( \vec{B} \left( \frac{\pi}{2} \right) = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle \). Thus its equation is
\[ \frac{1}{\sqrt{2}} (x - 0) + \frac{1}{\sqrt{2}} \left( z - \frac{\pi}{2} \right) = 0 \]

Multiplying each side by \( \sqrt{2} \) gives
\[ (x - 0) + \left( z - \frac{\pi}{2} \right) = 0 \]

or
\[ x + z = \frac{\pi}{2} \]

Make sure you have read, studied and understood what was done above before attempting the problems.

### 2.4.3 Problems

Do # 11, 13, 37, 39, 43 at the end of 10.3 in your book