1.4 Numerical Methods: The Approximation Method of Euler

1.4.1 Introduction

When an ODE cannot be solved, many numerical methods exist to approximate the solution of this ODE. We could devote an entire class to studying some of these methods. In this section, we review the oldest technique, originally devised by Euler. This method is called Euler’s method. It is based on the idea that the tangent line to the function \( y = f(x) \) at \( x = a \) is very close to the function \( f(x) \) for values of \( x \) close to \( a \) and can therefore be used as a replacement.

The method we describe in this section applies to first order ODEs.

1.4.2 Description of the Method and Examples

Let us start with the general IVP

\[
\begin{align*}
\frac{dy}{dx} &= f(x, y) \\
y(x_0) &= y_0
\end{align*}
\]  

(1.4)

Let us also assume that the conditions which guarantee existence and uniqueness of a solution are satisfied.

Suppose we want to approximate the solution to (1.4) near \( x = x_0 \), say when \( x = x_1 \). The exact value is \( y(x_1) \) but we can’t compute this value since we don’t know \( y(x) \), the solution to equation (1.4). However, we know \( x_0 \) and \( y_0 \) since they are given as the initial condition. We also know \( x_1 \), this is the x-value where we want to approximate the solution. Remembering the properties of tangent lines and also looking at figure 1.6, we know that the tangent line to the solution remains close to the solution near \( (x_0, y_0) \). If we call \( y_1 \) the point on the tangent line with x-coordinate \( x_1 \) then we see that as long as \( x_1 \) is close enough to \( x_0 \), \( y_1 \) will be a good approximation for \( y(x_1) \) and we will write \( y(x_1) \approx y_1 \). On the other hand, if \( x_1 \) is not close to \( x_0 \) then \( y_1 \) will not be a good approximation for \( y(x_1) \). We now show how to find \( y_1 \).

Going back to equation (1.4), we see that the slope of the solution at \( (x_0, y_0) \) is \( f(x_0, y_0) \) hence the equation of the line tangent to the solution at \( (x_0, y_0) \) is

\[
y = y_0 + f(x_0, y_0) (x - x_0)
\]

(1.5)

So, when \( x = x_1 \), we have that

\[
y_1 = y_0 + f(x_0, y_0) (x_1 - x_0)
\]

(1.6)

As we said earlier, if \( x_1 \) is close enough to \( x_0 \) then \( y(x_1) \approx y_1 \).

Now that we have approximated \( y(x_1) \), we can repeat the procedure to approximate \( y(x_2) \) if \( x_2 \) is close enough to \( x_1 \). We draw the line tangent to the solution at \( (x_1, y_1) \), its slope will be \( f(x_1, y_1) \) and the equation of the tangent
Figure 1.6: Solution and its Tangent Line at \((x_0, y_0)\)
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line will be $y = y_1 + f(x_1, y_1)(x - x_1)$ so when $x = x_2$, we see that $y_2 = y_1 + f(x_1, y_1)(x_2 - x_1)$.

We can repeat this procedure for a sequence of points $x_0$, $x_1$, $x_2$, ... For simplicity, let us assume these points are equally spaced, $h$ units apart. $h$ is called the step size and it is a small number (such as .1, .01, ...). Then, we can write

$$x_n = x_0 + nh, \quad \text{for } n = 0, 1, 2, ...$$

We wish to approximate $y(x_n)$ for $n = 1, 2, 3, ...$. We call the approximation $y_n$. From the computations done above, we see that

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$
$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1)$$

and in general, with $x_{n+1} = x_n + h$ for $n = 0, 1, 2, ...$ we have:

$$y_{n+1} = y_n + f(x_n, y_n)(x_{n+1} - x_n)$$

But $x_{n+1} - x_n = h$ hence we have

$$y_{n+1} = y_n + hf(x_n, y_n), \quad \text{for } n = 0, 1, 2, ... \quad (1.7)$$

This is known as Euler’s method. It can be used to approximate the solution to the IVP in equation 1.4. We illustrate it with an example.

**Example 1.4.1** Use Euler’s method with step size $h = 0.1$ to approximate the solutions to the IVP

$$\begin{cases} \frac{dy}{dx} &= x\sqrt{y} \quad \text{at the points } 1.1, 1.2, 1.3, 1.4, \text{ and } 1.5. \\ y(1) &= 4 \end{cases}$$

In our case, $x_0 = 1$, $y_0 = 4$, $h = 0.1$, and $f(x, y) = x\sqrt{y}$. hence, the recursive formula in equation 1.7 becomes $y_{n+1} = y_n + (0.1)x_n\sqrt{y_n}$.

- **When $n = 0$, we get**

  $$y_1 = y_0 + (0.1)x_0\sqrt{y_0}$$
  $$= 4 + (0.1)(1)\sqrt{4}$$
  $$= 4.2$$

  *Also note that $x_1 = x_0 + h = 1 + .1 = 1.1$.*

- **When $n = 1$, we get**

  $$y_2 = y_1 + (0.1)x_1\sqrt{y_1}$$
  $$= 4.2 + (0.1)(1.1)\sqrt{4.2}$$
  $$= 4.42543$$

  *Also note that $x_2 = x_1 + h = 1.1 + .1 = 1.2$.***
• When \( n = 2 \), we get

\[
y_3 = y_2 + (0.1) x_2 \sqrt{y_2}
\]

\[
= 4.42543 + (0.1)(1.2) \sqrt{4.42543}
\]

\[
= 4.67787
\]

Also note that \( x_3 = x_2 + h = 1.2 + .1 = 1.3 \)

• Continuing this way, we can build the table shown below. For comparison, we have included in the last column the exact values of the solution \( y(x) = \frac{(x^2 + 7)^2}{16} \).

<table>
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<th>( n )</th>
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<th>( y_n )</th>
<th>( y(x_n) )</th>
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<td>4</td>
<td>4</td>
</tr>
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<tr>
<td>5</td>
<td>1.5</td>
<td>5.27081</td>
<td>5.34766</td>
</tr>
</tbody>
</table>

Figure 1.7 shows the exact solution in red and Euler’s approximation in blue.

The next example shows how to approximate the solution at a point \( x \) not close to \( x_0 \). As figure 1.6 shows, the further \( x \) is from \( x_0 \), the further the tangent line will be from the solution, hence our approximation as we computed it above will not be a good approximation. A way around this is to introduce intermediate points, not too far from one another. In other words, we go from \( x_0 \) to \( x \) in several steps, not in just one step. If we wish to do it in \( N \) steps, then we set \( h = \frac{x - x_0}{N} \). This will create the points \( x_0, x_1, x_2, \ldots, x_N \) where \( x_{n+1} = x_n + h \) for \( n = 0, 1, 2, \ldots, N-1 \). Note that the point at which we want the approximation will always be \( x_N \). As before, we will call \( y_n \), the approximation for \( y(x_n) \). Our approximation will be \( y_N \). On one hand, it would seem that the larger \( N \) that is the smaller \( h \), the better our approximation should be. However, the larger \( N \) is, the more computations there will be. Since there are roundoff errors for each computation, the larger \( N \) is, the larger the accumulated roundoff error will be.

Example 1.4.2 Consider the IVP

\[
\begin{cases}
\frac{dy}{dx} = y \\ y(0) = 1
\end{cases}
\]  

(1.8)

Use Euler’s method to approximate the solution at \( x = 1 \) using 1) 1 step, 2) 2 steps, and 3) 4 steps.

Let us first remark that the solution to this equation is \( y(x) = e^x \) hence, we will
Figure 1.7: Solution: red
            Euler’s Approximation: blue
be approximating $e^1 = e$. Using the same notation as in the previous explanation and example, we see that $f(x, y) = y$, $x_0 = 0$ and $y_0 = 1$. The recursive formula for Euler’s method is

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$= y_n + hy_n$$

$$= (1 + h) y_n$$

1. When $N = 1$

   In this case, there are no intermediate points. We just have the starting point $x_0 = 0$ and the end point $x_1 = 1$. Hence, $h = \frac{x_1 - x_0}{N} = \frac{1 - 0}{1} = 1$. Euler’s formula gives $y_1 = (1 + 1) 1 = 2$ which is not a very good approximation of $e = 2.71828...$

2. When $N = 2$. This creates the points $x_0, x_1, x_2$ where $x_0 = 0$, $x_2 = 1$ and $h = \frac{1 - 0}{2} = .5$. It follows that $x_1 = 0.5$. Euler formula gives

$$y_1 = (1 + h) y_0$$

$$= (1.5)(1)$$

$$= 1.5$$

and

$$y_2 = (1 + h) y_1$$

$$= 1.5(1.5)$$

$$= 2.25$$

which is an improvement, but still not a very good approximation for $e$.

3. When $N = 4$. This creates the points $x_0, x_1, x_2, x_3, x_4 = 1$. As before, $h = 0.25$. It follows that $x_1 = 0.25$, $x_2 = 0.5$, and $x_3 = 0.75$. Euler’s formula gives

$$y_1 = (1 + h) y_0$$

$$= (1.25)(y_0)$$

$$= 1.25$$

also

$$y_2 = (1.25) y_1$$

$$= (1.25)(1.25)$$

$$= 1.5625$$

also

$$y_3 = (1.25) y_2$$

$$= (1.25)(1.5625)$$

$$= 1.9531$$
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and

\[ y_4 = (1.25) y_3 \]
\[ = (1.25) 1.9531 \]
\[ = 2.4414 \]

which is better than the previous approximation.

1.4.3 Exercises

Do # 1, 3, 5 at the end of 1.4 in your book.
Bibliography

