4.5 Basis and Dimension of a Vector Space

In the section on spanning sets and linear independence, we were trying to understand what the elements of a vector space looked like by studying how they could be generated. We learned that some subsets of a vector space could generate the entire vector space. Such subsets were called spanning sets. Other subsets did not generate the entire space, but their span was still a subspace of the underlying vector space. In some cases, the number of vectors in such a set was redundant in the sense that one or more of the vectors could be removed, without changing the span of the set. In other cases, there was not a unique way to generate some vectors in the space. In this section, we want to make this process of generating all the elements of a vector space more reliable, more efficient.

4.5.1 Basis of a Vector Space

Definition 297 *Let V denote a vector space and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ a subset of V. S is called a* ***basis*** *for V if the following is true:*

1. *S spans V.*
2. *S is linearly independent.*

This definition tells us that a basis has to contain enough vectors to generate the entire vector space. But it does not contain too many. In other words, if we removed one of the vectors, it would no longer generate the space.

A basis is the vector space generalization of a coordinate system in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Example 298 *We have already seen that the set $S = \{\mathbf{e}_1, \mathbf{e}_2\}$ where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ was a spanning set of $\mathbb{R}^2$. It is also linearly independent for the only solution of the vector equation $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 = 0$ is the trivial solution. Therefore, S is a basis for $\mathbb{R}^2$. It is called the* ***standard basis*** *for $\mathbb{R}^2$. These vectors also have a special name. $(1, 0)$ is $\mathbf{i}$ and $(0, 1)$ is $\mathbf{j}$.*

Example 299 *Similarly, the standard basis for $\mathbb{R}^3$ is the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$. These vectors also have a special name. They are $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ respectively.*

Example 300 *Prove that $S = \left\{1, x, x^2\right\}$ is a basis for P_2, the set of polynomials of degree less than or equal to 2.*
We need to prove that S spans P_2 and is linearly independent.

- *S spans P_2. We already did this in the section on spanning sets. A typical polynomial of degree less than or equal to 2 is $ax^2 + bx + c$.*
- *S is linearly independent. Here, we need to show that the only solution to $a(1) + bx + cx^2 = 0$ (where 0 is the zero polynomial) is $a = b = c = 0$.*

> *From algebra, we remember that two polynomials are equal if and only if their corresponding coefficients are equal. The zero polynomial has all its coefficients equal to zero. So,* $a(1) + bx + cx^2 = 0$ *if and only if* $a = 0$, $b = 0$, $c = 0$. *Which proves that* S *is linearly independent.*

We will see more examples shortly.

The next theorem outlines an important difference between a basis and a spanning set. Any vector in a vector space can be represented in a unique way as a linear combination of the vectors of a basis..

Theorem 301 *Let* V *denote a vector space and* $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ *a basis of* V. *Every vector in* V *can be written in a unique way as a linear combination of vectors in* S.

Proof. *Since* S *is a basis, we know that it spans* V. *If* $\mathbf{v} \in V$, *then there exists scalars* $c_1, c_2, ..., c_n$ *such that* $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + ... + c_n\mathbf{u}_n$. *Suppose there is another way to write* $\mathbf{v}$. *That is, there exist scalars* $d_1, d_2, ..., d_n$ *such that* $\mathbf{v} = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + ... + d_n\mathbf{u}_n$. *Then,* $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + ... + c_n\mathbf{u}_n = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + ... + d_n\mathbf{u}_n$. *In other words,* $(c_1 - d_1)\mathbf{u}_1 + (c_2 - d_2)\mathbf{u}_2 + ... + (c_n - d_n)\mathbf{u}_n = 0$. *Since* S *is a basis, it must be linearly independent. The unique solution to* $(c_1 - d_1)\mathbf{u}_1 + (c_2 - d_2)\mathbf{u}_2 + ... + (c_n - d_n)\mathbf{u}_n = 0$ *must be the trivial solution. It follows that* $c_i - d_i = 0$ *for* $i = 1, 2, ..., n$ *in other words* $c_i = d_i$ *for* $i = 1, 2, ..., n$. *Therefore, the two representations of* $\mathbf{v}$ *are the same.* ■

Remark 302 *We say that any vector* $\mathbf{v}$ *of* V *has a unique representation with respect to the basis* S. *The scalars used in the linear representation are called the coordinates of the vector. For example, the vector* (x, y) *can be represented in the basis* $\{(1,0), (0,1)\}$ *by the linear combination* $(x, y) = x(1,0) + y(0,1)$. *Thus,* x *and* y *are the coordinates of this vector (we knew that!).*

Definition 303 *If* V *is a vector space, and* $B = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ *is an ordered basis of* V, *then we know that every vector* $\mathbf{v}$ *of* V *can be expressed as a linear combination of the vectors in* S *in a unique way. In others words, there exists unique scalars* $c_1, c_2, ..., c_n$ *such that* $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + ... + c_n\mathbf{u}_n$. *These scalars are called the* ***coordinates*** *of* v *relative to the ordered basis* B.

Remark 304 *The term "ordered basis" simply means that the order in which we list the elements is important. Indeed it is since each coordinate is with respect to one of the vector in the basis. We know that in* $\mathbb{R}^2$, $(2,3)$ *is not the same as* $(3,2)$.

Example 305 *What are the coordinates of* $(1,2,3)$ *with respect to the basis* $\{(1,1,0), (0,1,1), (1,1,1)\}$*?*
One can indeed verify that this set is a basis for $\mathbb{R}^3$. *Finding the coordinates of* $(1,2,3)$ *with respect to this new basis amounts to finding the numbers* (a, b, c) *such that* $a(1,1,0) + b(0,1,1) + c(1,1,1) = (1,2,3)$. *This amounts to solving*

the system $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. *The solution is*

$$\begin{aligned} \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

The next theorem, deals with the number of vectors the basis of a given vector space can have. We will state the theorem without proof.

Theorem 306 *Let V denote a vector space and $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ a basis of V.*

1. *Any subset of V containing more than n vectors must be dependent.*
2. *Any subset of V containing less than n vectors cannot span V.*

Proof. *We prove each part separately.*

1. *Consider $W = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$ a subset of V where $r > n$. We must show that W in dependent. Since S is a basis, we can writ each $\mathbf{v}_i$ in term of elements in S. More specifically, there exists constants c_{ij} with $1 \leq i \leq r$ and $1 \leq j \leq n$ such that $\mathbf{v}_i = c_{i1}\mathbf{u}_1 + c_{i2}\mathbf{u}_2 + ... + c_{in}\mathbf{u}_n$. Consider the linear combination*

$$\sum_{j=1}^{r} d_j \mathbf{v}_j = \sum_{j=1}^{r} d_j \left(c_{j1}\mathbf{u}_1 + c_{j2}\mathbf{u}_2 + ... + c_{jn}\mathbf{u}_n \right) = 0$$

So, we must solve $\begin{cases} d_1c_{11} + d_1c_{12} + ... + d_1c_{1n} = 0 \\ d_2c_{21} + d_2c_{22} + ... + d_2c_{2n} = 0 \\ \vdots \\ d_rc_{r1} + d_rc_{r2} + ... + d_rc_{rn} = 0 \end{cases}$ *where the unknowns are $d_1, d_2, ..., d_r$. Since we have more unknowns than equations, we are guaranteed that this homogeneous system will have a nontrivial solution. Thus W is dependent.*

2. *Consider $W = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$ a subset of V where $r < n$. We must show that W does not span V. We do it by contradiction. We assume it does span V and show this would imply that S is dependent. Suppose that there exists constants c_{ij} with $1 \leq i \leq n$ and $1 \leq j \leq r$ such that $\mathbf{u}_i = c_{i1}\mathbf{v}_1 + c_{i2}\mathbf{v}_2 + ... + c_{ir}\mathbf{v}_r$. Consider the linear combination*

$$\sum_{j=1}^{n} d_j \mathbf{u}_j = \sum_{j=1}^{r} d_j \left(c_{j1}\mathbf{v}_1 + c_{j2}\mathbf{v}_2 + ... + c_{jr}\mathbf{v}_r \right) = 0$$

So, we must solve $\begin{cases} d_1c_{11} + d_1c_{12} + ... + d_1c_{1r} = 0 \\ d_2c_{21} + d_2c_{22} + ... + d_2c_{2r} = 0 \\ \vdots \\ d_nc_{n1} + d_nc_{n2} + ... + d_nc_{nr} = 0 \end{cases}$ *where the unknowns are* $d_1, d_2, ..., d_n$. *Since we have more unknowns than equations, we are guaranteed that this homogeneous system will have a nontrivial solution. Thus* S *would be dependent. But it can't be since it is a basis.*

■

Corollary 307 *Let* V *denote a vector space. If* V *has a basis with* n *elements, then all the bases of* V *will have* n *elements.*
Proof. *Assume that* S_1 *is a basis of* V *with* n *elements and* S_2 *is another basis with* m *elements. We need to show that* $m = n$. *Since* S_1 *is a basis,* S_2 *being also a basis implies that* $m \leq n$. *If we had* $m > n$, *by the theorem,* S_2 *would be dependent, hence not a basis. Similarly, since* S_2 *is a basis,* S_1 *being also a basis implies that* $n \leq m$. *The only way we can have* $m \leq n$ *and* $n \leq m$ *is if* $m = n$. ■

4.5.2 Dimension of a Vector Space

All the bases of a vector space must have the same number of elements. This common number of elements has a name.

Definition 308 *Let* V *denote a vector space. Suppose a basis of* V *has* n *vectors (therefore all bases will have* n *vectors).* n *is called the* ***dimension*** *of* V. *We write* $\dim(V) = n$.

Remark 309 n *can be any integer.*

Definition 310 *A vector space* V *is said to be* ***finite-dimensional*** *if there exists a finite subset of* V *which is a basis of* V. *If no such finite subset exists, then* V *is said to be* ***infinite-dimensional***.

Example 311 *We have seen, and will see more examples of finite-dimensional vector spaces. Some examples of infinite-dimensional vector spaces include* $F(-\infty, \infty)$, $C(-\infty, \infty)$, $C^m(-\infty, \infty)$.

Remark 312 *If* V *is just the vector space consisting of* $\{0\}$, *then we say that* $\dim(V) = 0$.

It is very important, when working with a vector space, to know whether its dimension is finite or infinite. Many nice things happen when the dimension is finite. The next theorem is such an example.

Theorem 313 *Let* V *denote a vector space such that* $\dim(V) = n < \infty$. *Let* $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ *be a subset of* V.

1. *If S spans V, then S is also linearly independent hence a basis for V.*

2. *If S is linearly independent, then S also spans V hence is a basis for V.*

This theorem says that in a finite dimensional space, for a set with as many elements as the dimension of the space to be a basis, it is enough if one of the two conditions for being a basis is satisfied.

4.5.3 Examples

Standard Basis of Known Spaces and Their Dimension

We look at some of the better known vector spaces under the standard operations, their standard bases, and their dimension.

Example 314 $\mathbb{R}^2$*, the set of all ordered pairs (x, y) where x and y are in $\mathbb{R}$. We have already seen that the standard basis for $\mathbb{R}^2$was $\{(1,0),(0,1)\}$. This basis has 2 elements, therefore,* $\dim\left(\mathbb{R}^2\right) = 2$.

Example 315 $\mathbb{R}^3$*, the set of all ordered triples (x, y, z) where x, y and z are in $\mathbb{R}$. Similarly, the standard basis for $\mathbb{R}^3$ is $\{(1,0,0),(0,1,0),(0,0,1)\}$. This basis has 3 elements, therefore,* $\dim\left(\mathbb{R}^3\right) = 3$.

Example 316 $\mathbb{R}^n$*, the set of all ordered n-tuples $(x_1, x_2, ..., x_n)$ where $x_1, x_2, ..., x_n$ are in $\mathbb{R}$. Similarly, the standard basis for $\mathbb{R}^n$ is $\{(1,0,0,...,0),(0,1,0,...,0),...,(0,0,...,0,1)\}$. This basis has n elements, therefore* $\dim(\mathbb{R}^n) = n$.

Example 317 P_2*, the set of polynomials of degree less than or equal to 2. We have already seen that the standard basis for P_2 was $\left\{1, x, x^2\right\}$. This basis has 3 elements, therefore* $\dim(P_2) = 3$.

Example 318 P_3*, the set of polynomials of degree less than or equal to 3. Similarly, the standard basis for P_3 is $\left\{1, x, x^2, x^3\right\}$. This basis has 4 elements, therefore* $\dim(P_3) = 4$.

Example 319 P_n*, the set of polynomials of degree less than or equal to n. Similarly, the standard basis for P_n is $\left\{1, x, x^2, ..., x^n\right\}$. This basis has $n+1$ elements, therefore* $\dim(P_n) = n + 1$.

Example 320 $M_{3,2}$*, the set of 2×3 matrices. The user will check as an exercise that a basis for $M_{3,2}$ is* $\left\{\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\\0&1\end{bmatrix}\right\}$. *This is the standard basis for $M_{3,2}$. This basis has 6 elements, therefore* $\dim(M_{3,2}) = 6$.

Example 321 $M_{m,n}$*, the set of $m \times n$ matrices. The standard basis for $M_{m,n}$ is the set $\{B_{ij} | i = 1, 2, ..., m$ and $j = 1, 2, ..., n\}$ where B_{ij} is the $m \times n$ matrix whose entries are all zeros, except for the ij entry which is 1. This set has mn elements, therefore* $\dim(M_{m,n}) = mn$

Example 322 *P, the set of all polynomials. This space does not have a finite dimension, in other words* $\dim(P) = \infty$. *To see this, assume that it has finite dimension, say* $\dim(P) = n$. *Let S be a basis for P. S has n elements. Let m be the highest degree of the polynomials which appear in S. Then, x^{m+1} cannot be obtained by linear combination of elements of S which contradicts the fact that S is a basis.*

Example 323 *Show that $B = \{(1,1),(0,1\}$ is a basis for $\mathbb{R}^2$.*
We know that $\dim\left(\mathbb{R}^2\right) = 2$. *Since B has two elements, it is enough to show that B is either independent or that it spans $\mathbb{R}^2$. We prove that B is independent. For this, we need to show that the only solution to $a(1,1) + b(0,1) = (0,0)$ is $a = b = 0$. The coefficient matrix of the corresponding system is $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. This matrix is invertible (its determinant is 1) therefore, the system has a unique solution. Since it is a homogeneous system, this unique solution is the trivial solution. Hence, B is linearly independent, therefore it is a basis by theorem 313.*

4.5.4 Dimension of Subspaces

In the examples that follow, given the description of a subspace, we have to find its dimension. For this, we need to find a basis for it.

Example 324 *The set of 2×2 symmetric matrices is a subspace of $M_{2,2}$. Find a basis for it and deduce its dimension.*
A typical 2×2 matrix is of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. However, for this matrix to be symmetric, we must have $b = c$. Therefore, a typical 2×2 symmetric matrix is of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Such a set can be spanned by $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$. This can be easily seen since $\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. For this set to be a basis, we must also prove that it is linearly independent. For this, we look at the solutions of the equation $a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This equation is equivalent to $\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ from which it follows that the only solution is $a = b = c = 0$. Therefore, $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ is a basis for the set of symmetric 2×2 symmetric matrices. Hence, the dimension of this set is 3.

Example 325 *Determine the dimension of the subspace W of $\mathbb{R}^3$ defined by $W = \{(d, c-d, c) \,|c \in \mathbb{R} \text{ and } d \in \mathbb{R}\}$.*
We notice that even though we are in $\mathbb{R}^3$, not all three coordinates of the typical

element of this subspace are independent. There is a pattern. We can write

$$\begin{aligned}(d, c-d, c) &= (0, c, c) + (d, -d, 0) \\ &= c(0,1,1) + d(1,-1,0)\end{aligned}$$

Thus, we see that the set $B = \{(0,1,1), (1,-1,0)\}$ *spans* W*. Is it a basis? We need to check if it is linearly dependent. For this, we solve the equation* $a(0,1,1) + b(1,-1,0) = (0,0,0)$*. This is equivalent to* $\begin{cases} b = 0 \\ a - b = 0 \\ a = 0 \end{cases}$*. We see that the only solution is* $a = b = 0$*. Therefore,* B *is independent, it follows that it is a basis for* W*. Hence,* $\dim(W) = 2$*.*

Example 326 *Determine the dimension of the subspace* W *of* $\mathbb{R}^3$ *defined by* $W = \{(x, y, 0) \,|\, x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$*. Give its geometric description.*
We notice that even though we are in $\mathbb{R}^3$*, not all three coordinates of the typical element of this subspace are independent. There is a pattern. We can write*

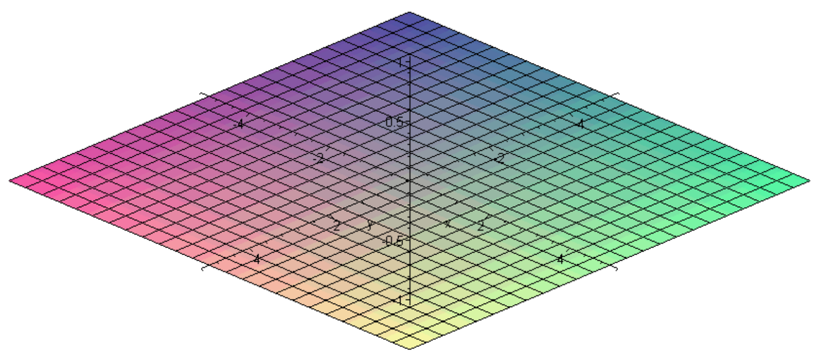
$$\begin{aligned}(x, y, 0) &= (x, 0, 0) + (0, y, 0) \\ &= x(1,0,0) + y(0,1,0)\end{aligned}$$

Thus, we see that $B = \{(1,0,0), (0,1,0)\}$ *spans* W*. It is also easy to check that it is linearly independent (left to the reader to do). Hence it is a basis for* W*. It follows that* $\dim(W) = 2$*.* W *is the set of vectors in* $\mathbb{R}^3$ *whose third coordinate is zero. These vectors only have an* x *and* y *coordinate. Therefore, the live in the x-y plane in* $\mathbb{R}^3$*. A picture of this plane is shown in figure 326.*

Next, we look at some example in which a spanning set of the subspace is given and we have to find its dimension. If the given spanning set is independent, it will be a basis and the dimension of the space will be the number of elements of the given set. Otherwise, we will have to eliminate from the spanning sets the vectors which are linear combinations of the other vectors in the given set, until we have a linearly independent set.

Example 327 *Determine the dimension of the subspace* W *of* $\mathbb{R}^4$ *spanned by* $S = \{(-1,2,5,0), (3,0,1,-2), (-5,4,9,2)\}$*.*
We must determine if S *is linearly independent. Let* $\mathbf{v}_1 = (-1,2,5,0)$*,* $\mathbf{v}_2 = (3,0,1,-2)$ *and* $\mathbf{v}_3 = (-5,4,9,2)$*. Then, it is easy to see that* $\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$*, thus proving* S *is linearly dependent. By a theorem studied before, we know we can remove* $\mathbf{v}_3$ *from* S *to obtain* $S_1 = (\mathbf{v}_1, \mathbf{v}_2)$ *and* S_1 *will span the same set as* S*. If* S_1 *is linearly independent, we are done. For this, we look at the equation* $a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{0}$*. It is equivalent to the system*

$$\begin{cases} -a + 3b = 0 \\ 2a = 0 \\ 5a + 9b = 0 \\ -2a + 2b = 0 \end{cases}$$


1
0.5
-4
-2
4
2
2
y
x
2
-0.5
4
4
-1

The only solution to this system is $a = b = 0$*. It follows that* S_1 *is independent, hence a basis for* W*. Hence,* $\dim(W) = 2$*.*

Remark 328 *Suppose we fail to see that* $\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$*, which was critical for doing this problem. What can we do? It's easy. Set up the system to see if the given set is independent. The system is* $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = 0$*. This system is equivalent to*

$$\begin{cases} -a + 3b - 5c = 0 \\ 2a + 4c = 0 \\ 5a + b + 9c = 0 \\ -2b + 2c = 0 \end{cases}$$

The corresponding augmented matrix is

$$\begin{bmatrix} -1 & 3 & -5 & 0 \\ 2 & 0 & 4 & 0 \\ 5 & 1 & 9 & 0 \\ 0 & -2 & 2 & 0 \end{bmatrix}$$

Reducing it with Gaussian elimination produces

$$\begin{bmatrix} -1 & 3 & -5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system is

$$\begin{cases} -a + 3b - 5c = 0 \\ b - c = 0 \end{cases}$$

The solutions are

$$\begin{cases} a = -2c \\ b = c \end{cases}$$

If we write the solution in parametric form, we obtain

$$\begin{cases} a = -2t \\ b = t \\ c = t \end{cases}$$

for any real number t*. Letting* $t = 1$*, we obtain* $a = -2$*,* $b = c = 1$*. Hence, the equation* $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = 0$ *becomes* $-2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 0$ *which is equivalent to*

$$\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$$

Example 329 *We proved in an earlier section that the solution set of a homogeneous system formed a vector space. In this example, we illustrate how to find* *its dimension and a basis for it. Determine a basis and the dimension of the solution space of*

$$\begin{cases} 2x_1 + 2x_2 - x_3 + x_5 = 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0 \\ x_1 + x_2 - 2x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 0 \end{cases}$$

We begin by solving the system. Its augmented matrix is

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & | & 0 \\ -1 & -1 & 2 & -3 & 1 & | & 0 \\ 1 & 1 & -2 & 0 & -1 & | & 0 \\ 0 & 0 & 1 & 1 & 1 & | & 0 \end{bmatrix}$$

Its reduced row-echelon form is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 1 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Thus its solutions are

$$\begin{cases} x_1 + x_2 + x_5 = 0 \\ x_3 + x_5 = 0 \\ x_4 = 0 \end{cases}$$

If we write the solution in parametric form, we get

$$\begin{cases} x_1 = -s - t \\ x_2 = s \\ x_3 = -t \\ x_4 = 0 \\ x_5 = t \end{cases}$$

So, we have

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} &= \begin{bmatrix} -s \\ s \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ -t \\ 0 \\ t \end{bmatrix} \\ &= s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Thus, the vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. *Since they are also linearly independent (check!), they form a basis. So, the dimension of the solution space is* 2.

We finish this section with a few very important theorems we will give without proof.

Theorem 330 *Let S be a nonempty set of vectors in V.*

1. *If S is linearly independent and $\mathbf{v} \in V$ but $\mathbf{v} \notin span\,(S)$ then $S \cup \{\mathbf{v}\}$ is also linearly independent.*

2. *If $\mathbf{v} \in S$ and $\mathbf{v}$ is a linear combination of elements of S then $span\,(S) = span\,(S - \{\mathbf{v}\})$*

Corollary 331 *Let S be a finite set of vectors of a finite-dimensonal vector space V.*

1. *If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

2. *If S is linearly independent but does not span V, then S can be enlarged to a basis for V by inserting appropriate vectors in S.*

Proof. *We prove each part separately.*

1. *If S is not a basis, then it is not independent. It means some vectors of S are linear combination of other vectors in S. By part 2 of the theorem, such vectors can be removed. This way, we can remove all the vectors of S which are linear combination of other vectors. When none such vectors are left, S will be linearly independent hence a basis.*

2. *Here, we use part 1 of the theorem. If $span\,(S) \neq V$, we can pick a vector in V not in $span\,(S)$ and add it to S. S will still be linearly independent. We continue to add vectors until $span\,(S) = V$.*

■

Theorem 332 *If W is any subspace of a finite-dimensional vector space V then* $\dim(W) \leq \dim(V)$. *Moreover, if* $\dim(W) = \dim(V)$ *then* $W = V$.

4.5.5 Summary

- Know and understand the definition of a basis for a vector space.
- Know what the dimension of a vector space is.
- Know what the coordinates of a vector relative to a given basis are.
- Given a set of vectors in a vector space, be able to tell if that set is a basis for the vector space.
- Know the standard basis for common vector spaces such as $\mathbb{R}^n$, M_{nn}, P_n for every positive integer n.
- Be able to find the basis of subspaces given the description of a subspace.
- Be able to find the coordinates of any vector relative to a given basis.
- Be able to find the dimension of a vector space.
- Know and understand the difference between a finite-dimensional and infinite dimensional vector space.

4.5.6 Problems

Exercise 333 *Let V be a vector space, and S a subset of V containing n vectors. What can be said about* $\dim(V)$ *if we know that S spans V?*

Exercise 334 *Let V be a vector space, and S a subset of V containing n vectors. What can be said about* $\dim(V)$ *if we know that S is linearly independent?*

Exercise 335 *Let V be a vector space of dimension n. Can a subset S of V containing less than n elements span V?*

Exercise 336 *Let V be a vector space of dimension n. Can a subset S of V containing less than n elements be dependent? If yes, is it always dependent?*

Exercise 337 *Same question for independent.*

Exercise 338 *Let V be a vector space, and S a subset of V containing n vectors. If S is linearly independent, will any subset of S be linearly independent? why?*

Exercise 339 *Let V be a vector space, and S a subset of V containing n vectors. If S is linearly dependent, will any subset of S be linearly dependent? why?*

Exercise 340 *What is the dimension of $C(-\infty, \infty)$ and why?*

Exercise 341 *Do # 1, 2, 3, 4, 6, 7, 8, 10, 12, 19, 21, 23, 24, 30, 32 on pages 263 - 265*