4.3 Linear Combinations and Spanning Sets

In the previous section, we looked at conditions under which a subset $W$ of a vector space $V$ was itself a vector space. In the next three section, we look at the following problem. If $W$ is not a vector space, how can we build a vector space from it? Once we answer that, we will try to find the most efficient way of doing it. We begin with some important definitions.

4.3.1 Linear Combinations

Definition 272 Let $V$ denote a vector space, and $v \in V$. We say that $v$ is a linear combination of the vectors $u_1, u_2, ..., u_n$ if $v$ can be written in the form

$$v = c_1 u_1 + c_2 u_2 + \ldots + c_n u_n = \sum_{i=1}^{n} c_i u_i$$

where $c_1, c_2, ..., c_n$ are scalar.

Because $V$ is a vector space, we know that given $u_1, u_2, ..., u_n$ in $V$, the linear combination $\sum_{i=1}^{n} c_i u_i$ is also in $V$. A more difficult question is: given $v \in V$ and $u_1, u_2, ..., u_n$ in $V$, can we find $c_1, c_2, ..., c_n$ such that $v = c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$?

To try to answer this question, we first look at some example to familiarize ourselves with this concept. In the examples.

Example 273 Let $\mathbb{R}^2$ be the underlying vector space. Is $v = (1, 2)$ a linear combination of the vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$?

It is easy to see that $v = e_1 + 2e_2$. In fact, if $v = (x, y)$, then $v = xe_1 + ye_2$. So, any vector in $\mathbb{R}^2$ can be represented as a linear combination of $e_1$ and $e_2$.

Example 274 Let $\mathbb{R}^2$ be the underlying vector space. Is $v = (1, 2)$ a linear combination of $u_1 = (1, 3)$ and $u_2 = (4, 1)$?

This time, it is a little bit more difficult. We must find scalars $c_1$ and $c_2$ such that $v = c_1 u_1 + c_2 u_2$. This amounts to solving the system

$$\begin{cases} c_1 + 4c_2 = 1 \\ 3c_1 + c_2 = 2 \end{cases}$$

Using any method learned in this class, we find that the system has a unique solution

$$\begin{cases} c_1 = \frac{7}{11} \\ c_2 = \frac{1}{11} \end{cases}$$

Therefore, $v = \frac{7}{11} u_1 + \frac{1}{11} u_2$. 
Example 275 Let $\mathbb{R}^3$ be the underlying vector space. Is $v = (1, 4, 6)$ a linear combination of $u_1 = (1, 2, 3)$, $u_2 = (1, 1, 2)$ and $u_3 = (1, 3, 4)$?

We need to solve the system $c_1 u_1 + c_2 u_2 + c_3 u_3 = v$. The augmented matrix of the system is

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & 3 & 4 \\
3 & 2 & 4 & 6 \\
\end{bmatrix}
$$

We use Gaussian elimination to solve the system. Applying $(R_2 - rR_1) \rightarrow (R_2)$ and $(R_3 - 3R_1) \rightarrow (R_3)$, we obtain

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & -1 & 1 & 2 \\
0 & -1 & 1 & 3 \\
\end{bmatrix}
$$

Then, applying $(R_3 - R_2) \rightarrow (R_3)$, we obtain

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & -1 & 1 & 2 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

From the last line, we see that the system has no solution. So, $v$ cannot be expressed as a linear combination of $u_1$, $u_2$ and $u_3$.

In general, given $v \in V$ and $u_1, u_2, ..., u_n$ in $V$, determining if $v$ is a linear combination of the vectors $u_1, u_2, ..., u_n$ is simply a matter of solving the system of linear equations $v = c_1 u_1 + c_2 u_2 + ... + c_n u_n$. We know that such systems can have no solution, a unique solution, or an infinite number of solutions. If the determinant of the coefficient matrix of the system is not zero, that is if the coefficient matrix is invertible, then the system has a unique solution. We will see later that it is important to know. However, if its determinant is zero, it does not mean there does not exist a linear combination. When the coefficient matrix of a system is not invertible, the system could have no solution, or an infinite number of solutions.

Now, we are starting to get answers to the questions we asked in the abstract. Given $u_1, u_2, ..., u_n$ in $V$, it is possible that not every vector of $V$ can be expressed as a linear combination of $u_1, u_2, ..., u_n$. When it is the case that every vector in $V$ can be expressed as a linear combination of $u_1, u_2, ..., u_n$, then the vectors $u_1, u_2, ..., u_n$ are special. It is what we study next.

4.3.2 Spanning Sets

Definition 276 Let $V$ denote a vector space and $S = \{u_1, u_2, ..., u_n\}$ a subset of $V$.

1. We say that $S$ is a spanning set of $V$ or that $S$ spans $V$ if for every vector $v$ in $V$, $v$ can be written as a linear combination of the vectors in $S$. 
2. The span of $S$, denoted $\text{span}(S)$, is the set of all linear combinations of vectors in $S$. In other words,

$$\text{span}(S) = \left\{ \sum_{i=1}^{n} c_i u_i | c_i \in \mathbb{R} \text{ and } u_i \in S \right\}$$

Before we look at specific examples, there are several important remarks to make and questions to ask.

**Remark 277**

1. Since $V$ is a vector space and $S \subseteq V$, it follows that $\text{span}(S)$ is a subset of $V$. Could $\text{span}(S)$ equal $V$?

2. In the case that $\text{span}(S)$ is a proper subset of $V$, what is $\text{span}(S)$?

3. Given $S$, how do we determine if $\text{span}(S) = V$?

4. Given $S$, how do we find $\text{span}(S)$?

We investigate the issues raised in the remark by first looking at some examples.

**Example 278** We saw earlier that $S = \{e_1, e_2\}$ where $e_1 = (1,0)$ and $e_2 = (0,1)$ was a spanning set for $\mathbb{R}^2$ because every vector $v = (x,y)$ in $\mathbb{R}^2$ could be written as a linear combination of $e_1$ and $e_2$ as follows: $v = xe_1 + ye_2$. It is called the standard spanning set of $\mathbb{R}^2$. There are others, as the next example suggests.

**Example 279** Show that $S = \{u_1, u_2\}$ where $u_1 = (1,1)$ and $u_2 = (0,1)$ spans $\mathbb{R}^2$

Here, we have to show that any element of $\mathbb{R}^2$ can be expressed as a linear combination of vectors of $S$. A typical element of $\mathbb{R}^2$ is of the form $v = (x,y)$. So, we have to show that the system $c_1 u_1 + c_2 u_2 = v$ always has at least one solution. This can be done several ways. First, we can try to solve the system using Gaussian elimination. In this case, there is a faster and easier way. The coefficient matrix of the system is

$$
\begin{bmatrix}
1 & 0 \\
1 & 1 \\
\end{bmatrix}
$$

The determinant of this matrix is not zero, this means that the system always has a unique solution. The system $c_1 u_1 + c_2 u_2 = v$ can always be solve, hence $\text{span}(S) = \mathbb{R}^2$.

**Example 280** Similarly, $S = \{e_1, e_2, e_3\}$ where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ is a spanning set for $\mathbb{R}^3$. It is called the standard spanning set of $\mathbb{R}^3$.

**Example 281** Let $P_2$ denote the set of polynomials of degree less than or equal to two. The set $S = \{1, x, x^2\}$ is a spanning set for $P_2$. A typical polynomial of $P_2$ is $ax^2 + bx + c$, it is easy to see how this can be written as a linear combination of elements of $S$. It is called the standard spanning set of $P_2$. 
Example 282  Does \( S = \{(2,1,-2), (-2,-1,2), (4,2,-4)\} \) span \( \mathbb{R}^3 \)?

A typical element of \( \mathbb{R}^3 \) is \((x,y,z)\). The question becomes: can the system \( c_1 (2,1,-2) + c_2 (-2,-1,2) + c_3 (4,2,-4) = (x,y,z) \) always be solved? Be careful, here the unknowns are \( c_1, c_2, \) and \( c_3, \) not \( x, y, z \). This system can also be written

\[
\begin{align*}
2c_1 - 2c_2 + 4c_3 &= x \\
c_1 - c_2 + 2c_3 &= y \\
-2c_1 + 2c_2 - 4c_3 &= z
\end{align*}
\]

The augmented matrix of the system is

\[
\begin{bmatrix}
2 & -2 & 4 & x \\
1 & -1 & 2 & y \\
-2 & 2 & -4 & z
\end{bmatrix}
\]

First, we perform \( R_2 \rightarrow R_2 - \frac{1}{2} R_1 \) → \( R_2 \) and \( R_3 + R_1 \rightarrow R_3 \) to obtain

\[
\begin{bmatrix}
2 & -2 & 4 & x \\
0 & 0 & -1 & \frac{1}{2} x + y \\
0 & 0 & 0 & \frac{1}{2} x + z
\end{bmatrix}
\]

For this system to have a solution, A vector of \( \mathbb{R}^3 \) must satisfy

\[
\begin{align*}
-\frac{1}{2} x + y &= 0 \\
x + z &= 0
\end{align*}
\]

Obviously, not every vector of \( \mathbb{R}^3 \) satisfies this. For example, \((1,1,1)\) does not. This means that not every vector of \( \mathbb{R}^3 \) can be written as a linear combination of vectors in \( S \). Thus \( \text{span}(S) \neq \mathbb{R}^3 \).

As noted earlier, \( \text{span}(S) \) is always a subset of the underlying vector space \( V \). If \( S \) is a spanning set, then \( \text{span}(S) = V \), otherwise, \( \text{span}(S) \) is a proper subset. But, even in this case, \( \text{span}(S) \) has important properties, as the next theorem tells us.

Theorem 283  Let \( V \) denote a vector space and \( S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \) a subset of \( V \).

1. \( \text{span}(S) \) is a subspace of \( V \).

2. \( \text{span}(S) \) is the smallest subspace of \( V \) containing \( S \).

Proof.  We prove each part separately.

1. We have already established that \( \text{span}(S) \) was a non-empty subset of \( V \).
   We simply need to show that it is closed under both operations. Let \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) be two elements of \( \text{span}(S) \), let \( c \) be a scalar.
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(a) Closure under addition. We must show that \( v_1 + v_2 \) is also in \( S \).

Since \( v_1 \in S \), there exist scalars \( c_1, c_2, \ldots, c_n \) such that \( v_1 = \sum_{i=1}^{n} c_i u_i \).

Similarly, \( v_2 = \sum_{i=1}^{n} d_i u_i \) for some scalars \( d_1, d_2, \ldots, d_n \). Therefore,

\[
  v_1 + v_2 = \sum_{i=1}^{n} c_i u_i + \sum_{i=1}^{n} d_i u_i = \sum_{i=1}^{n} (c_i + d_i) u_i
\]

And therefore, \( v_1 + v_2 \) is a linear combination of elements in \( S \).

(b) Closure under scalar multiplication. We must show that \( cv_1 \) is also in \( S \). Using the notation of the previous part of this proof, we have

\[
  cv_1 = c \sum_{i=1}^{n} c_i u_i = \sum_{i=1}^{n} cc_i u_i
\]

And therefore, \( cv_1 \) is a linear combination of elements in \( S \).

2. Let \( W \) be a subspace containing \( S \). We need to show that \( W \) contains \( \text{span}(S) \). This is obvious. Since \( W \) is a subspace, it is closed under addition and scalar multiplication. Therefore, if \( W \) contains \( S \), it must contain every linear combinations of elements of \( S \). But the set of all possible linear combinations of elements of \( S \) is precisely \( \text{span}(S) \).

There are still several unanswered questions. From the examples, we have seen that a space can have several spanning sets. Here are questions the reader should try to answer, especially as going through the next section.

1. If \( S \) is a spanning set of a vector space \( V \), and one adds another vector of \( V \) to \( S \), is the new set still a spanning set of \( V \)?

2. Same question in the case that we remove a vector from \( S \).

4.3.3 Problems

Do \# 7, 8, 9, 11, 13 23c, 23e on pages 239, 240.