The Inverse of a Matrix

Dr. Philippe B. Laval
Kennesaw State University
February 6, 2002

Abstract
This handout discusses the inverse of a matrix. The following topics are covered:

• Definitions
• Finding the inverse of a matrix using Gauss-Jordan elimination
• Properties of the inverse of a matrix
• Relationship between the inverse of a matrix and solving systems of equations

1 Definitions and Basic Properties

The inverse of a matrix is to matrix multiplication what the inverse of a number is for multiplication of real numbers. The notation used is similar. The inverse of a nonzero real number $a$ is denoted $\frac{1}{a}$ or $a^{-1}$. The inverse of a matrix $A$ is denoted $A^{-1}$.

Definition 1 (Inverse of a matrix) Let $A \in M_{n,n}$.

1. $A$ is said to be invertible or nonsingular if there exists a matrix $B$ such that $AB = BA = I_n$

2. The matrix $B$ in the above definition, when it exists, is denoted $A^{-1}$.

3. If $A$ does not have an inverse, then $A$ is said to be noninvertible or singular.

Remark 2 Some remarks on the inverse of a matrix.

1. Only square matrices can have an inverse.

2. Not every square matrix has an inverse.
3. The definition tells us that if the product of \( A \) by another matrix is the identity matrix, then the other matrix is an inverse of \( A \).

Given two matrices \( A \) and \( B \), to check if one is the inverse of the other, we simply have to check if

\[
AB = BA = I
\]

**Example 3** The two matrices

\[
A = \begin{bmatrix}
1 & 2 & -1 \\
2 & 1 & 0 \\
-1 & 1 & 2
\end{bmatrix}
\]

and

\[
B = \begin{bmatrix}
\frac{2}{9} & 5 & \frac{1}{9} \\
\frac{4}{9} & \frac{1}{9} & \frac{2}{9} \\
\frac{9}{1} & -\frac{1}{9} & \frac{9}{9}
\end{bmatrix}
\]

are inverses of each other because

\[
\begin{bmatrix}
1 & 2 & -1 \\
2 & 1 & 0 \\
-1 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
\frac{2}{9} & 5 & \frac{1}{9} \\
\frac{4}{9} & \frac{1}{9} & \frac{2}{9} \\
\frac{9}{1} & -\frac{1}{9} & \frac{9}{9}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\frac{2}{9} & 5 & \frac{1}{9} \\
\frac{4}{9} & \frac{1}{9} & \frac{2}{9} \\
\frac{9}{1} & -\frac{1}{9} & \frac{9}{9}
\end{bmatrix}
\begin{bmatrix}
1 & 2 & -1 \\
2 & 1 & 0 \\
-1 & 1 & 2
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Could \( A \) have another inverse? The next theorem gives us the answer.

**Theorem 4** For any nonsingular \( n \times n \) matrix \( A \) we have:

1. \( A^{-1} \) is unique
2. \( A^{-1} \) is also nonsingular and \( (A^{-1})^{-1} = A \)
3. If \( B \) is also an \( n \times n \) nonsingular matrix, then \( AB \) is nonsingular and \( (AB)^{-1} = B^{-1}A^{-1} \)

**Proof.** Let \( A, B \in M_{n,n} \), assume both matrices are nonsingular.
1. To prove that the inverse of $A$ is unique, we assume that $A$ has two inverses, and show they are the same. Let us assume that $A$ has two inverses we will call $M$ and $N$. In other words, we have

$$AM = MA = I$$
$$AN = NA = I$$

We wish to prove that $M = N$.

$$AM = I$$
$$N(AM) = NI$$
$$(NA)M = N$$
$$IM = N$$
$$M = N$$

2. Clearly, we have

$$A^{-1}A = AA^{-1} = I$$

That is, the product of $A^{-1}$ by $A$ is the identity matrix. By definition, $A$ is an inverse of $A^{-1}$. Since the inverse of a matrix is unique, it follows that $A$ is the inverse of $A^{-1}$. In other words, $(A^{-1})^{-1} = A$.

3. Since the inverse of a matrix is unique, once we find a matrix which satisfies the required condition, then we know we have found the inverse. To prove that $(AB)^{-1} = B^{-1}A^{-1}$ we simply have to prove that the product of $AB$ and $B^{-1}A^{-1}$ is the identity.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Remark 5 The theorem tells us the following:

1. The inverse of a matrix, when it exists, is unique. Keep in mind that the inverse may not exist.

2. If a matrix has an inverse, then its inverse also has an inverse, which is the original matrix.

3. When the product of two matrices is the identity matrix, then the two matrices are inverses of each other.
4. The product of two nonsingular matrices is also nonsingular. The inverse of a product is the product of the inverses, in reverse order.

Remark 6 Though matrix multiplication is not commutative, that is \( AB \neq BA \), it turns out that if both \( A \) and \( B \) are \( n \times n \) matrices and \( AB = I_n \) then it can be shown that \( BA = I_n \). In other words, to verify that \( A \) and \( B \) are inverses of each other, it is enough to check that \( AB = I_n \).

2 Finding the Inverse of a Nonsingular Matrix

Given an \( n \times n \) matrix \( A \) which is nonsingular (we assume it has an inverse), how do we find its inverse? Answering this question amounts to finding an \( n \times n \) matrix \( B \) satisfying

\[
AB = I
\]

where \( I \) is the identity matrix of the correct size. We introduce the following notation:

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
    b_{11} & b_{12} & \cdots & b_{1n} \\
    b_{21} & b_{22} & \cdots & b_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{n1} & b_{n2} & \cdots & b_{nn}
\end{bmatrix}
\]

Thus, solving \( AB = I \) amounts to finding the entries \((b_{ij})\). Also, we let \( B_j \) denote the \( j^{th} \) row of \( B \). In other words,

\[
B_1 = \begin{bmatrix}
    b_{11} \\
    b_{21} \\
    \vdots \\
    b_{n1}
\end{bmatrix}
\]

\[
B_2 = \begin{bmatrix}
    b_{12} \\
    b_{22} \\
    \vdots \\
    b_{n2}
\end{bmatrix}
\]

\[
B_n = \begin{bmatrix}
    b_{1n} \\
    b_{2n} \\
    \vdots \\
    b_{nn}
\end{bmatrix}
\]
Similarly, we let $I_j$ denote the $j^{th}$ row of $I$. So,

$$I_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Then, we see that solving the equation $AB = I$ is the same as solving

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

This matrix equation is equivalent to solving the $n$ systems $AB_j = I_j$ for $j = 1, 2, \ldots, n$.

We know how to solve each of these systems, for example using Gauss-Jordan elimination. Since the transformations involved in Gauss-Jordan elimination only depend on the coefficient matrix $A$, we will realize that we are repeating a lot of the work if we solve the $n$ systems. It is more efficient to do the following:

1. Form the matrix $[A : I]$ by adjoining $A$ and $I$. Note it will be an $n \times 2n$ matrix.

2. If possible, row reduce $A$ to $I$ using Gauss-Jordan elimination on the entire matrix $[A : I]$. The result will be the matrix $[I : A^{-1}]$. If this is not possible, then $A$ does not have an inverse.

We illustrate the procedure by doing some examples.

**Example 7** Find the inverse of $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$.

In other words, we want to find $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ such that $AB = I$. We
begin by adjoining the identity matrix to $A$ to obtain:

$$
\begin{bmatrix}
  2 & -17 & 11 & : & 1 & 0 & 0 \\
  -1 & 11 & -7 & : & 0 & 1 & 0 \\
  0 & 3 & -2 & : & 0 & 0 & 1
\end{bmatrix}
$$

We then row reduce $A$ to the identity matrix by performing Gauss-Jordan elimination to the whole matrix. The transformation $(E_2 + \frac{1}{2} E_1) \rightarrow (E_2)$ produces

$$
\begin{bmatrix}
  2 & -17 & 11 & : & 1 & 0 & 0 \\
  0 & \frac{5}{2} & -\frac{3}{2} & : & \frac{1}{2} & 1 & 0 \\
  0 & 3 & -2 & : & 0 & 0 & 1
\end{bmatrix}
$$

The transformation $(2E_2) \rightarrow (E_2)$ produces

$$
\begin{bmatrix}
  2 & -17 & 11 & : & 1 & 0 & 0 \\
  0 & 5 & -3 & : & 1 & 2 & 0 \\
  0 & 3 & -2 & : & 0 & 0 & 1
\end{bmatrix}
$$

The transformation $(E_3 - \frac{3}{5}E_2) \rightarrow (E_3)$ produces

$$
\begin{bmatrix}
  2 & -17 & 11 & : & 1 & 0 & 0 \\
  0 & 5 & -3 & : & 1 & 2 & 0 \\
  0 & 0 & -\frac{1}{5} & : & -\frac{3}{5} & -\frac{6}{5} & 1
\end{bmatrix}
$$

The transformation $(-5E_3) \rightarrow (E_3)$ produces

$$
\begin{bmatrix}
  2 & -17 & 11 & : & 1 & 0 & 0 \\
  0 & 5 & -3 & : & 1 & 2 & 0 \\
  0 & 0 & 1 & : & 3 & 6 & -5
\end{bmatrix}
$$

The transformation $(E_2 + 3E_3) \rightarrow (E_2)$ produces

$$
\begin{bmatrix}
  2 & -17 & 11 & : & 1 & 0 & 0 \\
  0 & 5 & 0 & : & 10 & 20 & -15 \\
  0 & 0 & 1 & : & 3 & 6 & -5
\end{bmatrix}
$$
The transformation \((1/5E_2) \rightarrow (E_2)\) produces
\[
\begin{bmatrix}
2 & -17 & 11 & : & 1 & 0 & 0 \\
0 & 1 & 0 & : & 2 & 4 & -3 \\
0 & 0 & 1 & : & 3 & 6 & -5 \\
\end{bmatrix}
\]

The transformation \((E_1 - 11E_3) \rightarrow (E_1)\) produces
\[
\begin{bmatrix}
2 & -17 & 0 & : & -32 & -66 & 55 \\
0 & 1 & 0 & : & 2 & 4 & -3 \\
0 & 0 & 1 & : & 3 & 6 & -5 \\
\end{bmatrix}
\]

The transformation \((E_1 + 17E_2) \rightarrow (E_1)\) produces
\[
\begin{bmatrix}
2 & 0 & 0 & : & 2 & 2 & 4 \\
0 & 1 & 0 & : & 2 & 4 & -3 \\
0 & 0 & 1 & : & 3 & 6 & -5 \\
\end{bmatrix}
\]

Finally, the transformation \((1/2E_1) \rightarrow (E_1)\) produces
\[
\begin{bmatrix}
1 & 0 & 0 & : & 1 & 1 & 2 \\
0 & 1 & 0 & : & 2 & 4 & -3 \\
0 & 0 & 1 & : & 3 & 6 & -5 \\
\end{bmatrix}
\]

So,
\[
A^{-1} = \begin{bmatrix}
1 & 1 & 2 \\
2 & 4 & -3 \\
3 & 6 & -5 \\
\end{bmatrix}
\]

Example 8 Find the inverse of \(B = \begin{bmatrix}
1 & -2 & 1 \\
2 & -3 & 0 \\
-1 & 3 & -3 \\
\end{bmatrix}\).

We begin by adjoining the identity matrix to \(B\). We obtain
\[
\begin{bmatrix}
1 & -2 & 1 & : & 1 & 0 & 0 \\
2 & -3 & 0 & : & 0 & 1 & 0 \\
-1 & 3 & -3 & : & 0 & 0 & 1 \\
\end{bmatrix}
\]
Next, we try to row reduce $B$ to the identity matrix by applying Gauss-Jordan elimination to the whole matrix. Performing $(E_2 - 2E_1) \rightarrow (E_2)$ produces

\[
\begin{bmatrix}
1 & -2 & 1 & : & 1 & 0 & 0 \\
0 & 1 & -2 & : & -2 & 1 & 0 \\
-1 & 3 & -3 & : & 0 & 0 & 1
\end{bmatrix}
\]

Performing $(E_3 + E_1) \rightarrow (E_3)$ produces

\[
\begin{bmatrix}
1 & -2 & 1 & : & 1 & 0 & 0 \\
0 & 1 & -2 & : & -2 & 1 & 0 \\
0 & 1 & -2 & : & 1 & 0 & 1
\end{bmatrix}
\]

The transformation $(E_3 - E_2) \rightarrow (E_3)$ produces

\[
\begin{bmatrix}
1 & -2 & 1 & : & 1 & 0 & 0 \\
0 & 1 & -2 & : & -2 & 1 & 0 \\
0 & 0 & 0 & : & 3 & -1 & 1
\end{bmatrix}
\]

We do not need to continue, the last row of what used to be $B$ consists entirely of 0’s. $B$ does not have an inverse. Another way of saying this is that $B$ is singular.

2.1 Special Case: $2 \times 2$ Matrices

This procedure of finding the inverse of a matrix can be used with any matrix. It is very straightforward and can be easily programmed into a computer. However, it is a little bit long as these two example will attest. For $2 \times 2$ matrices, there is a faster way to find the inverse.

**Proposition 9** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then,

\[
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

**Proof.** We simply verify that $AA^{-1} = I$

\[
AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{ad - bc} \\ \frac{1}{ad - bc} & 1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

\[
= \frac{1}{ad - bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

\[
= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & -ab + ab \\ dc - dc & -bc + ad \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
We see in particular that when \( ad - bc = 0 \), the matrix does not have an inverse. We will revisit the quantity \( ad - bc \) later. It is called the \textbf{determinant} of \( A \). The determinant of a matrix is an important quantity for matrices. Its value provides a lot of information about a matrix.

## 3 More Properties of Inverses

We already studied some properties of the inverse of a matrix, in section 1, theorem 4. We list a few more important properties.

\textbf{Theorem 10} \ If \( A \) is an \( n \times n \) invertible matrix, \( k \) is a positive integer and \( c \) is a scalar then \( A^k \), \( cA \) and \( A^t \) are invertible. Furthermore,

1. \( (A^k)^{-1} = (A^{-1})^k \)
2. \( (cA)^{-1} = \frac{1}{c}A^{-1} \)
3. \( (A^t)^{-1} = (A^{-1})^t \)

\textbf{Proof.}

1. \textit{Using the fact that} \( A^k = AA...A \) (\( k \) factors) \textit{and part 3 of theorem 4, we have}

\[
(A^k)^{-1} = (AA...A)^{-1} = A^{-1}A^{-1}...A^{-1} = (A^{-1})^k
\]

2. \textit{We prove this by multiplying the two matrices, and using the properties of matrix operations.}

\[
(cA)^{-1} \left( \frac{1}{c}A^{-1} \right) = \frac{1}{c}A^{-1}A = \frac{1}{c}I = I
\]

3. \textit{Left as an exercise.}

The above theorem allows us to extend the notation \( A^k \) to exponents that are negative integers for nonsingular matrices. If \( k \) is a positive integer, then \( A^k = AA...A \) (\( k \) factors) . \( A^{-k} = (A^{-1})^k \). This way, the parallel between real numbers and matrices is complete. The only thing we do not do for matrices is
to use the notation $\frac{1}{A}$. We will always use $A^{-1}$ when talking about the inverse of $A$.

Another important property real numbers have is the cancellation law. We have already mentioned the fact that matrices did not have this property. We now see a special case under which they have it.

**Theorem 11** If $C$ is an invertible matrix, then the following is true.

1. If $AC = BC$, then $A = B$. This is called the right cancellation property.
2. If $CA = CB$, then $A = B$. This is called the left cancellation property.

**Proof.** To prove part 1, we use the fact that $C$ is invertible

\[
AC = BC \Rightarrow (AC)C^{-1} = (BC)C^{-1} \\
\Rightarrow A(CC^{-1}) = B(CC^{-1}) \\
\Rightarrow AI = BI \\
\Rightarrow A = B
\]

Part 2 is proven the same way. □

**Remark 12**

1. We must list the right and left cancellation properties. Because matrix multiplication is not commutative, having one of them does not necessarily imply that the other one is also true.

2. When it comes to the cancellation property, matrices behave like real numbers. The cancellation property for real numbers says that $ac = bc \Rightarrow a = b$ if $c \neq 0$. This is the same as saying $ac = bc \Rightarrow a = b$ if $c$ has an inverse, because, for real numbers, only $0$ does not have an inverse. That is exactly what the cancellation property for matrices says.

4 Relationship Between the Inverse of a Matrix and Solving Systems of Linear Equations

We already know that a system of linear equations can be written as the matrix equation $Ax = b$ where $A$ is the coefficient matrix, $x$ is the column matrix containing the variables and $b$ is the column matrix of constants. If $A$ is an invertible matrix, we now know how to solve this equation.

**Theorem 13** If $A$ is an invertible matrix, then the system of linear equations $Ax = b$ has a unique solution. The solution is given by

\[ x = A^{-1}b \]
Proof. Using the fact that $A$ is nonsingular, and the properties of matrix operations, we obtain

$$Ax = b \Rightarrow A^{-1}(Ax) = A^{-1}b$$
$$\Rightarrow (A^{-1}A)x = A^{-1}b$$
$$\Rightarrow Ix = A^{-1}b$$
$$\Rightarrow x = A^{-1}b$$

Remark 14 1. If $A$ is not invertible, we can only say that the system does not have a unique solution. The system could have no solution, or an infinite number of solutions. We will learn later how to further study $A$ to determine which situation we are in.

2. The above theorem tells us that if the system is homogeneous, and $A$ is invertible, then it has a unique solution, the trivial solution.

3. Though in theory the above theorem gives us a way to compute the solution for a system of linear equation, this solution is not very practical. Finding the inverse of a matrix requires more work that solving a system of equation with either Gaussian or Gauss-Jordan elimination. Furthermore, once we have the inverse, we have to multiply $b$ by $A^{-1}$. The only times it is practical to use is if either $A^{-1}$ is already known, or several systems with the same matrix $A$ have to be solved. In this case, we do the work of finding $A^{-1}$ once, then use it for the remaining systems.

Example 15 Solve the systems

\[
\begin{pmatrix}
2x - 17y + 11z = 1 \\
-x + 11y - 7z = 2 \\
3y - 2z = 3
\end{pmatrix}, \quad \begin{pmatrix}
2x - 17y + 11z = 2 \\
-x + 11y - 7z = 1 \\
3y - 2z = 1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
2x - 17y + 11z = 5 \\
-x + 11y - 7z = 4 \\
3y - 2z = 6
\end{pmatrix}
\]

We see that these systems use the same coefficient matrix $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$.

We have already computed its inverse and found that $A^{-1} = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$.

Therefore, the solution of the first system is

\[
x = A^{-1}b = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
= \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}
\]
The solution of the second system is

\[ x = A^{-1}b \]
\[ = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \]
\[ = \begin{bmatrix} 5 \\ 5 \\ 7 \end{bmatrix} \]

The solution of the third system is

\[ x = A^{-1}b \]
\[ = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix} \]
\[ = \begin{bmatrix} 21 \\ 8 \\ 9 \end{bmatrix} \]

5 Problems

1. Using your knowledge of diagonal matrices and inverse matrices, find a general formula for the inverse of a diagonal matrix. Do all diagonal matrices have an inverse?

2. On page 77 - 79, do the following problems: 3, 5, 9, 11, 19, 25, 29, 37, 38, 40, 41, 43, 46, 51, 52, 53

3. Decide whether each statement below is True or False. Justify your answer by citing a theorem, or giving a counter example.

   (a) If \( A \) is invertible, then the system \( Ax = b \) is consistent.
   (b) If \( A \) is not invertible, then the system \( Ax = b \) is consistent.
   (c) If \( A \) is not invertible, then the system \( Ax = b \) is not consistent.
   (d) If \( A \) is not invertible, then the system \( Ax = 0 \) is consistent.
   (e) If \( A \) is not invertible, then the system \( Ax = 0 \) is not consistent.