4.6 Null Space, Column Space, Row Space

In applications of linear algebra, subspaces of $\mathbb{R}^n$ typically arise in one of two situations: 1) as the set of solutions of a linear homogeneous system or 2) as the set of all linear combinations of a given set of vectors. In this section, we will study, compare and contrast these two situations. We will finish the section with an introduction to linear transformations.

4.6.1 The Null Space of a Matrix

Definitions and Elementary Remarks and Examples

In previous section, we have already seen that the set of solutions of a homogeneous linear system formed a vector space (theorem 271). This space has a name.

Definition 342 *The* ***null space*** *of an $m \times n$ matrix A, denoted $Null\ A$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Written in set notation, we have*

$$Null\ A = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

Remark 343 *As noted earlier, this is a subspace of $\mathbb{R}^n$. In particular, the elements of $Null\ A$ are vectors in $\mathbb{R}^n$ if we are working with an $m \times n$ matrix.*

Remark 344 *We know that $Null\ A \neq \emptyset$ since it always contain $\mathbf{0}$, the trivial solution. The question is can it have nontrivial solutions?*

Example 345 *The elements of $Null\ A$ if A is 3×5 are vectors of $\mathbb{R}^5$.*

Example 346 *The elements of $Null\ A$ if A is 2×2 are vectors in $\mathbb{R}^2$.*

Example 347 *The elements of $Null\ A$ if A is 3×2 are vectors in $\mathbb{R}^2$.*

Example 348 *The elements of $Null\ A$ if A is 5×2 are vectors in $\mathbb{R}^2$.*

Remark 349 *The kind of elements $Null\ A$ contains (which vector space they belong to) depends only on the number of columns of A.*

We now look at specific examples and how to find the null space of a matrix.

Examples

Usually, when one is trying to find the null space of a matrix, one tries to find a basis for it. So, when asked to "find the null space" of a matrix, one is asked to find a basis for it. The examples below illustrate how to do this.

Example 350 *Find Null A for* $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

First, let us remark that the elements of Null A will be elements of $\mathbb{R}^5$*. We are finding the solution set of* $A\mathbf{x} = \mathbf{0}$*. The augmented matrix of the system is* $\begin{bmatrix} -3 & 6 & -1 & 1 & -7 & \vdots & 0 \\ 1 & -2 & 2 & 3 & -1 & \vdots & 0 \\ 2 & -4 & 5 & 8 & -4 & \vdots & 0 \end{bmatrix}$*, the reduced row-echelon form is:* $\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & \vdots & 0 \\ 0 & 0 & 1 & 2 & -2 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$.

The solutions of the system are $\begin{cases} x_1 = 2r + s - 3t \\ x_2 = r \\ x_3 = -2s + 2t \\ x_4 = s \\ x_5 = t \end{cases}$ *which can be written as*

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$. *Null A is the subspace spanned by* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *where* $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$. *It should be clear that this set is also linearly independent. So, it is a basis for Null A. Hence, Null A has dimension* 3 *and it is the subspace of* $\mathbb{R}^5$ *with basis* $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Example 351 *Find Null A for* $A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

First, let us remark that the elements of Null A will be elements of $\mathbb{R}^5$*. To find Null A, we solve the system* $A\mathbf{x} = \mathbf{0}$*. The augmented matrix is* $\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$.

Its reduced row-echelon form is $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$*. So, the solutions are*

$\begin{cases} x_1 + x_2 + x_5 = 0 \\ x_3 + x_5 = 0 \\ x_4 = 0 \end{cases}$ *that is* $\begin{bmatrix} x_1 = -t - s \\ x_2 = t \\ x_3 = -s \\ x_4 = 0 \\ x_5 = s \end{bmatrix}$ *which can be written as* $\mathbf{x} = s \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} +$ $t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. *Thus Null A is a subspace of* $\mathbb{R}^5$, *of dimension* 2 *with basis* $\left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Additional Theoretical Results

If should be clear to the reader that if A is invertible then $Null\ A = \{\mathbf{0}\}$. Indeed, if A is invertible, then $A\mathbf{x} = \mathbf{0}$ only has the trivial solution. We state it as a theorem.

Theorem 352 *If A is invertible then $Null\ A = \{\mathbf{0}\}$.*

In earlier chapters, we developed the technique of elementary row transformations to solve a system. In particular, we saw that performing elementary row operations did not change the solutions of linear systems. We state this result as a theorem.

Theorem 353 *Elementary row operations on a matrix A do not change Null A.*

Definition 354 *The* ***nullity*** *of a matrix A, denoted nullity (A) is the dimension of its null space.*

Example 355 *From the previous examples, we see that if* $A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ *then nullity* $(A) = 2$ *and if* $B = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$ *then nullity* $(B) = 3$.

4.6.2 Column Space and Row Space of a Matrix

Definitions and Elementary Concepts

Let us start by formalizing some terminology we have already used.

Definition 356 *Consider an $m \times n$ matrix* $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$.

1. *The vectors* $\mathbf{r}_1 = \begin{bmatrix} a_{11}, & a_{12}, & a_{13}, & \cdots, & a_{1n} \end{bmatrix}$, $\mathbf{r}_2 = \begin{bmatrix} a_{21}, & a_{22}, & a_{23}, & \cdots, & a_{2n} \end{bmatrix}$, ..., $\mathbf{r}_m = \begin{bmatrix} a_{m1}, & a_{m2}, & a_{m3}, & \cdots, & a_{mn} \end{bmatrix}$ *in* $\mathbb{R}^n$ *obtained from the rows of A are called the **row vectors** of A.*

2. *The vectors* $\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{bmatrix}$, ..., $\mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{mn} \end{bmatrix}$ *in* $\mathbb{R}^m$ *obtained from the columns of A are called the **column vectors** of A.*

We now turn to the main definitions of this section.

Definition 357 *Let A be an $m \times n$ matrix.*

1. *The subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the **row space** of A.*

2. *The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the **column space** of A.*

In this subsection, we will try to answer the following questions:

1. How does one find the row space of a matrix A?
2. How does one find the column space of a matrix A?
3. Is there a relationship between them?
4. Is there a relationship between them and the null space of a matrix A?
5. Since matrices are closely related to solving systems of linear equations, is there a relationship between the row space, column space, null space of a matrix A and the linear system $A\mathbf{x} = \mathbf{b}$?

With the notation already introduced, and letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ we see that we can write

$$A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + ... + x_n\mathbf{c}_n$$

Thus, solving $A\mathbf{x} = \mathbf{b}$ amounts to solving $x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + ... + x_n\mathbf{c}_n = \mathbf{b}$. So, we see that for the system to have a solution, $\mathbf{b}$ must be in the span of $\{\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_n\}$. We state this a a theorem.

Theorem 358 *A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.*

We now look at some important results about the column space and the row space of a matrix.

Theoretical Results

First, we state and prove a result similar to one we already derived for the null space.

Theorem 359 *Elementary row operations do not change the row space of a matrix A.*
Proof. *Suppose that A is $m \times n$ with rows $\mathbf{r}_1, \mathbf{r}_n, ..., \mathbf{r}_m$. Let B be obtained from A by one of the elementary row operations. Suppose that the rows of B are $\mathbf{r}'_1, \mathbf{r}'_n, ..., \mathbf{r}'_m$. We divide the proof according to the kind of elementary row transformation being applied. We do the proof by showing that every vector in the row space of B is in the row space of A and vice-versa. In other words, we prove inclusion both ways; a standard technique to show two sets are equal.*

- *Row interchange. This case is easy. A and B will still have the same row space since they will have the same rows.*
- *Replacing a row by a multiple of another or by itself plus a multiple of another. This can be generalized by saying that one or more of $\mathbf{r}'_i$ are linear combinations of the $\mathbf{r}_j$s. Thus the vectors $\mathbf{r}'_1, \mathbf{r}'_n, ..., \mathbf{r}'_m$ lie in the row space of A. Since a vector space is closed under linear combinations, any linear combination of $\mathbf{r}'_1, \mathbf{r}'_n, ..., \mathbf{r}'_m$ will also be in the row space of A. But that's precisely what the row space of B is, linear combinations of $\mathbf{r}'_1, \mathbf{r}'_n, ..., \mathbf{r}'_m$. Thus the row space of B is in the row space of A.*
- *Since every elementary row transformation has an inverse transformation, we can transform B into A. Using the same argument as above, we will prove that the row space of A in in the row space of B.*

■

Remark 360 *This result, unfortunately, does not apply to the column space of A. Only its row space is preserved under elementary row operations. However, we do have the following results: The next two theorems are given without proof.*

Theorem 361 *Let A and B be two matrices which are row equivalent (one is obtained from the other with elementary row operations).*

1. *A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.*
2. *A given set of column vectors of A form a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.*

The following theorem makes it easy to find a basis for the row and column space of a matrix. We will use it in the examples.

Theorem 362 *If a matrix R is in row-echelon form then:*

1. *The row vectors with the leading $1'$s form a basis for the row space of R.*
2. *The column vectors with the leading $1'$s of the row vectors form a basis for the column space of R.*

Let us make some remarks about this theorem.

Remark 363 *Combining theorems 361 and 362, we see that to find a basis for the row space of a matrix of a matrix, we put it in row-echelon form and extract the row vectors with a leading 1.*

Remark 364 *Since row elementary row operations do not preserve the column space, to find the column space of a matrix will be a little more difficult. But not too much. Let A be a matrix and B be the row-echelon form of A. If we call $\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_n$ the columns of A and $\mathbf{c}'_1, \mathbf{c}'_2, ..., \mathbf{c}'_n$ the columns of B. We look at the columns of B which have a leading 1 of the rows of B. A basis for the column space of A will be the vectors $\mathbf{c}_i$ for the values of i for which $\mathbf{c}'_i$ has a leading 1 of the rows of B. This sounds complicated but it is not. We will illustrate this with examples.*

Finally, we give a theorem which relates the null space of A and solutions of $A\mathbf{x} = \mathbf{b}$.

Theorem 365 *If $\mathbf{x}_0$ is any solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$ form a basis for the null space of A, then any solution of $A\mathbf{x} = \mathbf{b}$ can be written as*

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_r\mathbf{v}_r \tag{4.1}$$

and conversely, any vector written this way for any scalars $c_1, c_2, ..., c_r$ is a solution of $A\mathbf{x} = \mathbf{b}$.
Proof. *We prove both ways.*

- *Assume $\mathbf{x}_0$ is any solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$ form a basis for the null space of A. We must prove that any solution of $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_r\mathbf{v}_r$. If $\mathbf{x}$ is any solution of $A\mathbf{x} = \mathbf{b}$ then we have $A\mathbf{x}_0 = \mathbf{b}$ and $A\mathbf{x} = \mathbf{b}$ that is $A\mathbf{x}_0 = A\mathbf{x}$ or $A(\mathbf{x} - \mathbf{x}_0) = \mathbf{0}$. Thus, $\mathbf{x} - \mathbf{x}_0$ is in the null space of A, that is there exists scalars $c_1, c_2, ..., c_r$ such that $\mathbf{x} - \mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_r\mathbf{v}_r$ or $\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_r\mathbf{v}_r$.*
- *Conversely, suppose that for any scalars $c_1, c_2, ..., c_r$, $\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_r\mathbf{v}_r$, where $\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$ we must show that $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$.*

$$\begin{aligned} A\mathbf{x} &= (\mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_r\mathbf{v}_r) \\ &= A\mathbf{x}_0 + c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + ... + c_rA\mathbf{v}_r \\ &= \mathbf{b} + \mathbf{0} + \mathbf{0} + ... + \mathbf{0} \end{aligned}$$

since the $\mathbf{v}'_i s$ are a basis for $Null\ A$. Thus we see that $A\mathbf{x} = \mathbf{b}$.

■

We now see how these results help us finding the column space and the row space of a matrix.

Examples

We begin with very easy examples.

Example 366 *Consider the matrix in row-echelon form* $A=\begin{bmatrix}1&-2&5&0&3\\0&1&3&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}$.
Find a basis for the row space and column space.
Since the matrix is in row-echelon form, we can apply theorem 362 directly.

- *Row Space: A basis is the set of row vectors with a leading* 1*, that is* $\left\{\begin{bmatrix}1&-2&5&0&3\end{bmatrix},\begin{bmatrix}0&1&3&0&0\end{bmatrix},\begin{bmatrix}0&0&0&1&0\end{bmatrix}\right\}$*. Hence the row space has dimension* 3*.*

- *Column Space: A basis is* $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}$*. Hence the column space has dimension* 3*.*

Example 367 *Find bases for the row space and column space of* $A=\begin{bmatrix}1&-3&4&-2&5&4\\2&-6&9&-1&8&2\\2&-6&9&-1&9&7\\-1&3&-4&2&-5&-4\end{bmatrix}$.
We begin by finding the row-echelon form of A*. It is* $\begin{bmatrix}1&-3&4&-2&5&4\\0&0&1&3&-2&-6\\0&0&0&0&1&5\\0&0&0&0&0&0\end{bmatrix}$.

- *Row Space: A basis is the set of row vectors with a leading* 1*, that is* $\left\{\begin{bmatrix}1&-3&4&-2&5&4\end{bmatrix},\begin{bmatrix}0&0&1&3&-2&-6\end{bmatrix},\begin{bmatrix}0&0&0&0&1&5\end{bmatrix}\right\}$*. Hence the row space has dimension* 3*.*

- *Column Space: The columns with a leading* 1 *from the row vectors are* 1*,* 3*, and* 5*. Hence, a basis for the column space are columns* 1*,* 3*, and* 5 *from the original matrix that is* $\left\{\begin{bmatrix}1\\2\\2\\-1\end{bmatrix},\begin{bmatrix}4\\9\\9\\-4\end{bmatrix},\begin{bmatrix}5\\8\\9\\-5\end{bmatrix}\right\}$*. Hence the column space has dimension* 3*.*

Remark 368 *In the two examples above, we see that the row space and column space have the same dimension. This was not an accident. We will see that indeed they do have the same dimension.*

4.6.3 Application to Linear Transformations

A transformation is the linear algebra term of a concept studied in calculus which was then called a function. Another difference is that a transformation works on vectors. Its input values are vectors, its output values are also vectors. The input and output values may not be from the same vector space. Like functions, a transformation assigns to each input value a unique output value. In this section, we only look at a special kind of transformations called linear transformations. We now give a precise mathematical definition.

Definition 369 (Linear Transformation) *A* ***linear transformation*** *T from a vector space V to another vector space W is a rule which assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that:*

1. *$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}$ and $\mathbf{v}$ in V.*
2. *$T(c\mathbf{u}) = cT(\mathbf{u})$ for every $\mathbf{u}$ in V and every scalar c.*

Remark 370 *The two conditions which have to be satisfied is what make the transformation a linear transformation.*

Remark 371 *If T is a transformation from V to W, we often write $T : V \to W$.*

Let us look at some linear transformations we already know.

Example 372 *Let A be an $m \times n$ matrix. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. It is easy to see that T is a linear transformation from the properties of matrices.*

$$\begin{aligned} T(\mathbf{x}+\mathbf{x}') &= A(\mathbf{x}+\mathbf{x}') \\ &= A\mathbf{x} + A\mathbf{x}' \\ &= T(\mathbf{x}) + T(\mathbf{x}') \end{aligned}$$

and

$$\begin{aligned} T(c\mathbf{x}) &= A(c\mathbf{x}) \\ &= cA\mathbf{x} \\ &= cT(\mathbf{x}) \end{aligned}$$

Example 373 *Let V be the vector space of real valued functions with continuous first derivatives defined on an interval $[a,b]$ and let W be the space of continuous functions on $[a,b]$. Consider the transformation $D : V \to W$ defined by $D(f) = f'$. It is easy to see that D is indeed a linear transformation.*

$$\begin{aligned} D(f+g) &= (f+g)' \\ &= f' + g' \text{ (sum rule for differentiation)} \\ &= D(f) + D(g) \end{aligned}$$

and

$$\begin{aligned} D(cf) &= (cf)' \\ &= cf' \\ &= cD(f) \end{aligned}$$

When dealing with linear transformation, there are two concepts of importance, defined below.

Definition 374 *Let $T : V \to W$ be a linear transformation.*

1. *The **kernel** of T, denoted $\ker(T)$ is the set of vectors $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{0}$.*
2. *The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.*

Example 375 *If the linear transformation is as in example 373, then $\ker(D)$ is the set of functions f such that $D(f) = 0$ that is $f' = 0$. So, it is the set of constant functions.*

Remark 376 *If T is as in example 372, then the kernel of T is simply the null space of A and the range of T is simply the column space of A.*

Example 377 *Let $T : \mathbb{R}^6 \to \mathbb{R}^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$. Find the kernel and the range of T.*

- *Kernel of T: As noticed in the remark, finding the kernel of T amounts to finding the null space of A. So, we need to solve the system $A\mathbf{x} = \mathbf{0}$. The augmented matrix of the system is $\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 0 \\ 2 & -6 & 9 & -1 & 8 & 2 & 0 \\ 2 & -6 & 9 & -1 & 9 & 7 & 0 \\ -1 & 3 & -4 & 2 & -5 & -4 & 0 \end{bmatrix}$. Its reduced row-echelon form is $\begin{bmatrix} 1 & -3 & 0 & -14 & 0 & -37 & 0 \\ 0 & 0 & 1 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Thus, the solutions are $\begin{cases} x_1 - 3x_2 - 14x_4 - 37x_6 = 0 \\ x_3 + 3x_4 + 4x_6 = 0 \\ x_5 + 5x_6 = 0 \end{cases}$, or written in parametric form: $\begin{cases} x_1 = 3s + 14t + 37u \\ x_2 = s \\ x_3 = -3t - 4u \\ x_4 = t \\ x_5 = -5u \\ x_6 = u \end{cases}$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 14 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} +$*

$u\begin{bmatrix} 37 \\ 0 \\ -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$. *So, we see that the dimension of* $\ker(T)$ *is* 3.

- *Range of* T*: As noticed in the remark, it is the column space of* A*. We already computed it in a previous example and found that it was the subspace of* $\mathbb{R}^4$ *with basis* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \right\}$. *Hence the range of* T *has dimension* 3.

4.6.4 Problems

1. Do # 1, 2, 3a, 3b, 3c, 6, 7, 8, 9, 11, 14, 16 on pages 277, 278
2. Suppose that $T : V \to W$ is a linear transformation.
   (a) Prove that $\ker(T)$ is a subspace of V.
   (b) Prove that the range of T is a subspace of W.