4.3 Explicit Finite Difference Method for the Heat Equation

4.3.1 Goals
Several techniques exist to solve PDEs numerically. In this section, we present the technique known as finite differences, and apply it to solve the one-dimensional heat equation. With this technique, the PDE is replaced by algebraic equations which then have to be solved. We will be solving an IBVP of the form

\[
\begin{align*}
\text{PDE} & \quad u_t = \alpha u_{xx} \quad 0 < x < L \quad 0 < t < \infty \\
\text{BC} & \quad u(0,t) = T_1 \\
& \quad u(L,t) = T_2 \quad 0 < t < \infty \\
\text{IC} & \quad u(x,0) = f(x) \quad 0 \leq x \leq L
\end{align*}
\]

We will also consider slight variations of this problem such as changing the BCs.

4.3.2 Outline of the Method
When solving the one-dimensional heat equation, it is important to understand that the solution \( u(x,t) \) is a function of two variables. The domain of \( u(x,t) \) is a subset of the xt-plane. More specifically, the domain is \( \{ (x,t) : 0 \leq x \leq L \text{ and } 0 \leq t < \infty \} \) or \( \{ (x,t) : (x,t) \in [0,L] \times [0,\infty) \} \). When we find an exact solution also known as a closed-form solution, we find \( u(x,t) \) for every pair \( (x,t) \) in \( [0,L] \times [0,\infty) \). However, when we solve a PDE numerically, the continuous xt-plane (infinitely many points) is replaced by a finite set of points, the PDE is replaced by algebraic equations and the algebraic equations are solved at each point. More specifically, we follow the following steps:

1. We discretize the domain. During this operation, the continuous xt-plane is replaced by a discrete domain, a finite set of points.

2. We discretize the PDE. During this operation, all the partial derivatives which appear in the PDE are approximated by finite differences. During this process, the PDE is replaced by algebraic equations.

3. We discretize the BCs and the IC. This step is similar to step 2.

4. We solve the algebraic equations obtained in steps 2 and 3 at the points obtained in step 1.

We now describe each step in detail.

4.3.3 Discretization of the Domain
The first step is to discretize the domain (xt-plane). We divide the interval \([0,L]\) of the x-axis into \( n + 1 \) points equally spaced. We label these points \( x_0, x_1, x_2, \ldots, x_n \). Let \( h = \Delta x = \frac{L}{n} \) denote the distance between two consecutive points.
Then we have \( x_i = ih \). Similarly, we divide the interval \((0, \infty)\) of the \( t \)-axis into points equally spaced. We label these points \( t_0, t_1, t_2, \ldots \). Let \( k = \Delta t \) be the distance between two consecutive points. Then we have \( t_j = jk \). We will solve the PDE at the points \((x_i, t_j)\) for \( i = 0, 1, \ldots, n \), \( j = 0, 1, 2, \ldots \). We also use the following notation:

\[
\begin{align*}
    u(x_i, t_j) & = u_{i,j} \\
    u(x_i + h, t_j) & = u_{i+1,j} \\
    u(x_i - h, t_j) & = u_{i-1,j} \\
    u(x_i, t_j + k) & = u_{i,j+1} \\
    u(x_i, t_j - k) & = u_{i,j-1}
\end{align*}
\]

**Example 165** The point \((0, 0)\) is obtained with \( i = 0 \) and \( j = 0 \) hence \((0, 0) = (x_0, t_0)\). The value of the function \( u \) at this point, \( u(0, 0) \) will be denoted \( u_{0,0} \).

**Example 166** The point \((L, 0)\) is obtained with \( i = n \) and \( j = 0 \) hence \((L, 0) = (x_n, t_0)\). The value of the function \( u \) at this point, \( u(L, 0) \) will be denoted \( u_{n,0} \).

**Example 167** The family of points \((x, 0)\) is replaced on the discrete domain by the family of points \((x_i, 0)\) and is obtained with \( j = 0 \) and for any \( i \) hence \((x, 0) = (x_i, t_0)\). The value of the function \( u \) at this point, \( u(x_i, 0) \) will be denoted \( u_{i,0} \).

**Example 168** The family of points \((0, t)\) is replaced on the discrete domain by the family of points \((0, t_j)\) and is obtained with \( i = 0 \) and for any \( j \) hence \((0, t) = (x_0, t_j)\). The value of the function \( u \) at this point, \( u(x_0, t_j) \) will be denoted \( u_{0,j} \).

**Example 169** The family of points \((L, t)\) is replaced on the discrete domain by the family of points \((L, t_j)\) and is obtained with \( i = n \) and for any \( j \) hence \((L, t) = (x_n, t_j)\). The value of the function \( u \) at this point, \( u(x_n, t_j) \) will be denoted \( u_{n,j} \).

All this amounts to putting a grid on the domain as shown in figure 4.2. The points \( u_{i,j} \) are represented on the grid. Keep in mind that the \( i \) index represents the spatial variable \( x \) and the \( j \) index represents the time variable \( t \). It is very important to understand the position of these points with respect to each other. It will help understand how the heat equation can be solved for one value of \( t \) at a time.

### 4.3.4 Approximating Derivatives

The reader will recall from section 4.2 the following formulas to approximate derivatives:

1. **Forward difference approximation of** \( u_x(x, t) \)

   \[
   u_x(x, t) \approx \frac{u(x + h, t) - u(x, t)}{h}
   \]
2. Backward difference approximation of $u_x(x, t)$

$$u_x(x, t) \approx \frac{u(x, t) - u(x - h, t)}{h}$$

3. Central difference approximation of $u_x(x, t)$

$$u_x(x, t) \approx \frac{u(x + h, t) - u(x - h, t)}{2h}$$

4. Central difference approximation of $u_{xx}(x, t)$

$$u_{xx}(x, t) \approx \frac{u(x + h, t) - 2u(x, t) + u(x - h, t)}{h^2}$$

5. Forward difference approximation of $u_t(x, t)$

$$u_t(x, t) \approx \frac{u(x, t + k) - u(x, t)}{k}$$

6. Backward difference approximation of $u_t(x, t)$

$$u_t(x, t) \approx \frac{u(x, t) - u(x, t - k)}{k}$$

7. Central difference approximation of $u_t(x, t)$

$$u_t(x, t) \approx \frac{u(x, t + k) - u(x, t - k)}{2k}$$
8. Central difference approximation of \( u_{tt}(x, t) \)

\[
u_{tt}(x, t) \approx \frac{u(x, t + k) - 2u(x, t) + u(x, t - k)}{k^2}
\]

Using the notation introduced above, we write:

1. Forward difference approximation of \( u_x(x_i, t_j) \)

\[
u_x(x_i, t_j) \approx \frac{u_{i+1,j} - u_{i,j}}{h}
\]

2. Backward difference approximation of \( u_x(x_i, t_j) \)

\[
u_x(x_i, t_j) \approx \frac{u_{i,j} - u_{i-1,j}}{h}
\]

3. Central difference approximation of \( u_x(x_i, t_j) \)

\[
u_x(x_i, t_j) \approx \frac{u_{i+1,j} - u_{i-1,j}}{2h}
\]

4. Central difference approximation of \( u_{xx}(x_i, t_j) \)

\[
u_{xx}(x_i, t_j) \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}
\]

5. Forward difference approximation of \( u_t(x_i, t_j) \)

\[
u_t(x_i, t_j) \approx \frac{u_{i,j+1} - u_{i,j}}{k}
\]

6. Backward difference approximation of \( u_t(x_i, t_j) \)

\[
u_t(x_i, t_j) \approx \frac{u_{i,j} - u_{i,j-1}}{k}
\]

7. Central difference approximation of \( u_t(x_i, t_j) \)

\[
u_t(x_i, t_j) \approx \frac{u_{i,j+1} - u_{i,j-1}}{2k}
\]

8. Central difference approximation of \( u_{tt}(x_i, t_j) \)

\[
u_{tt}(x_i, t_j) \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}
\]

Remark 170: \( h \) and \( k \) are called the step size.
4.3.5 Finite-Difference approximation of the Heat Equation

We now have everything we need to replace the PDE, the BCs and the IC. We look at some examples.

A First Example

Consider the problem

\[
\begin{align*}
\text{PDE} & : u_t = u_{xx} & 0 < x < 1 & 0 < t < \infty \\
\text{BC} & : u(0,t) = 0 & & 0 < t < \infty \\
\text{IC} & : u(x,0) = \sin \pi x + x & 0 \leq x \leq 1
\end{align*}
\]

The solution is a function of \(x\) and \(t\). Its domain is the \(xt\)-plane. It can be shown that the exact solution is \(u(x,t) = x + e^{-\pi^2 t} \sin \pi x\). Its graph is shown in figure 4.3.5 If we intersect the surface with planes \(t = C\) for some constant \(C\), that is planes parallel to the \(xu\)-coordinate plane, then we obtain a curve which is the temperature of the rod at time \(t = C\). Figure 4.3.5 shows the temperature of the rod at time \(t = .02\) (red), \(t = .1\) (green) and \(t = .5\) (blue). The initial condition is shown in black.
Discretization of the BCs As noticed earlier, the point \((0, t)\) is replaced on the discrete domain by \((0, t_j) = (x_0, t_j)\) and the value of the function \(u\) at that point is denoted \(u_{0,j}\). Similarly, \((1, t)\) is replaced on the discrete domain by \((x_n, t_j)\) and the value of the function \(u\) at that point is denoted \(u_{n,j}\). Hence, the BCs becomes \(u_{0,j} = 0\) and \(u_{n,j} = 1\) for \(j = 1, 2, 3, \ldots\). Note in particular that this means \(u_{i,j}\) is unknown only for \(i = 1, 2, 3, \ldots, n - 1\).

Discretization of the IC The point \((x, 0)\) is replaced on the discrete domain by \((x_i, 0) = (x_i, t_0)\) and the value of the function at that point is denoted \(u_{i,0}\). Hence, the IC becomes \(u_{i,0} = \sin \pi (i h) + i h\) for \(i = 0, 1, 2, \ldots, n\).

Discretization of the PDE We have several choices to approximate derivatives.

- We will use the forward difference approximation for \(u_t\) and the central difference approximation for \(u_{xx}\). Using the formulas we derived above, we get
  \[
  \frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}
  \]
  for \(i = 1, 2, \ldots, n - 1\) and \(j = 0, 1, 2, \ldots\).

- Note that this results in a large number of equations to solve. We have an equation for each value of \(i\) and \(j\). However, in this case, things are not too difficult. Solving for \(u_{i,j+1}\) gives us
  \[
  u_{i,j+1} = \frac{k}{h^2} \left( u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right) + u_{i,j}
  \]
  \[
  u_{i,j+1} = \left(1 - 2s\right) u_{i,j} + s \left( u_{i+1,j} + u_{i-1,j} \right)
  \]
  where
  \[
  s = \frac{k}{h^2}
  \]
  Note that all the expressions on the right are at one time level, the expression on the left is at the next time level as shown in figure 4.3. The implication is that once we know \(u\) for one time level, using equation 4.19 we can find \(u\) for the next time level, and so on. Of course, this implies that we must know \(u\) for one time level, which we do. Since we know \(u_{i,0}\) for every \(i\), we can get \(u_{i,1}\) for \(i = 1, 2, \ldots, n - 1\). Using the boundary condition, we can also get \(u_{0,1}\) and \(u_{n,1}\). We can then solve for the next time level and repeat the procedure. This is called an explicit-type marching process. Figure 4.4 illustrates this process. The black dots indicate the grid points where the solution is known either through the initial condition (dots on the \(x\)-axis) or the dots on the vertical boundaries (boundary conditions). They key to the success of this method is that we know the solution for the level \(j = 0\) (initial condition) as well as at the levels \(i = 0\) and \(i = n\) (boundary conditions).
Figure 4.3: The value of $u$ at a time level depend on the values of $u$ at the previous time level.

Figure 4.4: Forward Different Method for the Heat Equation.
A Second Example

Consider the problem

\[
\begin{align*}
\text{PDE} & \quad u_t = u_{xx} \quad 0 < x < 1 \quad 0 < t < \infty \\
\text{BC} & \quad u_x (0, t) = 0 \quad 0 < t < \infty \\
\text{IC} & \quad u (1, t) = 1 \\
& \quad u (x, 0) = \sin \pi x + x \quad 0 \leq x \leq 1
\end{align*}
\]

This is a function of \(x\) and \(t\). Its domain is the \(xt\)-plane. This is the same problem as the previous one, except for one of the boundary conditions. We are no longer given the solution on the left boundary. We can derive it from what we are given by discretizing \(u_x\) using a forward difference approximation:

\[
 u_x (x, y) \approx \frac{u_{i+1,j} - u_{i,j}}{h}. \quad \text{At the point } (0, t) \text{ for any } t, \ i = 0. \quad \text{Thus, } u_x (0, t) \approx \frac{u_{1,j} - u_{0,j}}{h}. \quad \text{The condition } u_x (0, t) = 0 \text{ gives us } \frac{u_{1,j} - u_{0,j}}{h} = 0 \iff u_{1,j} - u_{0,j} = 0 \text{ or } u_{0,j} = u_{1,j}. \quad \text{Since we can find } u_{1,j} \text{ for any } j \text{ using our discretize PDE, we can also find } u_{0,j} \text{ for any } j.
\]

### 4.3.6 A Stability Criterion

The step size in both \(x\) and \(t\) has to be small enough for our finite differences to be a good approximation of the derivatives. However, there is a trade-off. The smaller the step size, the more computations will be involved. In addition, for the explicit method, another factor must be taken into consideration. The step size in the \(t\) direction must be much smaller than the step size in the \(x\) direction. Though these depend on the PDE as well as the BCs, it can be shown that if \(0 < s \leq \frac{1}{2}\) for the heat equation, then the explicit method is stable, that is it gives reasonable approximations to the exact solution. If \(s > \frac{1}{2}\), the explicit method is unstable.

Looking at our first example, where \(s = \frac{k}{h^2}\), we see that \(k = sh^2\). So, if \(s = \frac{1}{2}\), the largest value \(s\) can have for the scheme to be stable, and \(h = .1\), then \(k = \frac{1}{2} \cdot \frac{1}{100} = \frac{1}{200} = 0.005\). So, we see that \(k\) is much smaller than \(h\).

### 4.3.7 Problems

1. Considering the second example, how would you have discretized the boundary conditions if they had been as shown below?

\[
\begin{align*}
u (0, t) &= 0 \\
u_x (1, t) &= 1
\end{align*}
\]
2. Find the finite-difference solution of the problem

\[
\begin{align*}
\text{PDE} & \quad u_t = u_{xx} & 0 < x < 1 & \quad 0 < t < \infty \\
\text{BC} & \quad u_x (0, t) = 0 & \quad 0 < t < \infty \\
\text{IC} & \quad u (x, 0) = \sin \pi x & 0 \leq x \leq 1
\end{align*}
\]

by the explicit method. Plot the solution using \( h = .1 \) and \( k = .005 \) for the first four time levels \( (j = 0, 1, 2, 3) \).

3. Solve the above problem (#2) using the separation of variables method and compare the values you got in problem #2 with those the exact solution would give.

4. Same problem as 2. but replace the BC at \( x = 1 \) with \( u_x (1, t) = - (u (1, t) - 1) \).

5. Find the finite-difference solution of the problem

\[
\begin{align*}
\text{PDE} & \quad u_t = u_{xx} & 0 < x < 1 & \quad 0 < t < \infty \\
\text{BC} & \quad u_x (0, t) = A (t) & \quad 0 < t < \infty \\
\text{IC} & \quad u (x, 0) = f (x) & 0 \leq x \leq 1
\end{align*}
\]

6. Find the finite-difference solution of the problem

\[
\begin{align*}
\text{PDE} & \quad u_t = u_{xx} & 0 < x < 1 & \quad 0 < t < \infty \\
\text{BC} & \quad u_x (0, t) = u (0, t) & \quad 0 < t < \infty \\
\text{IC} & \quad u (x, 0) = 1 & 0 \leq x \leq 1
\end{align*}
\]

7. Redo problems 2, 5 and 6 with the PDE \( u_t = \alpha u_{xx} \).

8. Redo problem 7 with the PDE \( u_t = \alpha u_{xx} + g (x) \) in other words, the heat equation with source.