2.4 Mixed or Robin Boundary Conditions

2.4.1 Goal
Learn how to solve a IBVP with homogeneous mixed boundary conditions and in the process, learn how to handle eigenvalues when they do not have a 'nice' formula.

2.4.2 An example with Mixed Boundary Conditions
The examples we did in the previous section with Dirichlet, Neumann, or periodic homogeneous boundary conditions all led to eigenvalue problems with eigenvalues which could be found easily and had a nice formula. We see here that it is not always the case.

Let us solve
\[ \begin{aligned}
\text{PDE} & : \quad u_t = u_{xx} \quad \quad 0 < x < 1 \quad 0 < t < \infty \\
\text{BC} & : \quad u_x (0,t) - u (0,t) = 0 \quad \quad 0 < t < \infty \\
\text{IC} & : \quad u (1,t) = 0 \quad \quad 0 < t < \infty \\
\end{aligned} \tag{2.47} \]

We recognize here the one-dimensional heat equation for a thin rod of length 1. The initial temperature distribution is given by the function \( \phi (x) \). The second boundary condition says that the right end of the rod is maintained at 0\(^\circ\). The first boundary condition is equivalent to \( u_x (0,t) = u (0,t) \). It is a mixed boundary condition. The general form (see section 1.3) is \( u_x (0,t) = \frac{h}{k} \left( u (0,t) - g_1 (t) \right) \) and it indicates that the temperature of the surrounding medium at the left end is \( g_1 (t) \). \( h \) and \( k \) are positive constants. Here we have \( u_x (0,t) = u (0,t) \) which can be rewritten as \( u_x (0,t) = 1 \left( u (0,t) - 0 \right) \) so it indicates that the temperature of the surrounding medium is 0 and the constants \( h \) and \( k \) are equal since \( \frac{h}{k} = 1 \).

Step 1: Transform the PDE in two ODEs
We seek a nontrivial solution of the form \( u (x,t) = X (x) T (t) \). Proceeding as in the previous sections, we transform the PDE in two ODEs. We get:
\[ \begin{aligned}
T' + \lambda k T & = 0 \\
X'' + \lambda X & = 0 \\
\end{aligned} \]

Step 2: Use the BCs to Start Solving the two ODEs
The solution of the first ODE is
\[ T (t) = C_1 e^{-\lambda kt} \]
2.4. MIXED OR ROBIN BOUNDARY CONDITIONS

Since \( u(x,t) = X(x)T(t) \) must satisfy the boundary conditions, we must have \( u_x(0,t) = X'(0)T(t) = u(0,t) = X(0)T(t) \) for every \( t > 0 \). Since we are looking for a nontrivial solution, \( T(t) \neq 0 \) hence dividing by \( T(t) \) implies that \( X'(0) = X(0) \). Similarly \( u(1,t) = X(1)T(t) = 0 \). To have a nontrivial solution, we must keep \( X(1) = 0 \). Thus \( X(x) \) must satisfy the following boundary value problem (BVP)

\[
\begin{align*}
X'' + \lambda X &= 0 \\
X'(0) &= X(0) \\
X(1) &= 0
\end{align*}
\]  

(2.48)

We know from the previous section, that \( \lambda \) is real and \( \lambda \geq 0 \) (see homework). So, we only need to look at the following cases:

- **\( \lambda = 0 \)**: This corresponds to the trivial solution (see homework).

- **\( \lambda > 0 \)**: In this case, the general solution of \( X'' + \lambda X = 0 \) is \( X(x) = C_2 \sin \sqrt{\lambda}x + C_3 \cos \sqrt{\lambda}x \). Hence, \( X'(x) = C_2 \sqrt{\lambda} \cos \sqrt{\lambda}x - C_3 \sqrt{\lambda} \sin \sqrt{\lambda}x \).

  The first boundary condition gives \( X(0) = C_3 = X'(0) = C_2 \sqrt{\lambda} \) so \( C_2 \sqrt{\lambda} = C_3 \). Thus \( X(x) = C_2 \sin \sqrt{\lambda}x + C_2 \sqrt{\lambda} \cos \sqrt{\lambda}x \). The second boundary condition gives us \( 0 = X(1) = C_2 \sin \sqrt{\lambda} + C_2 \sqrt{\lambda} \cos \sqrt{\lambda} \).

  So, \( 0 = C_2 (\sin \sqrt{\lambda} + \sqrt{\lambda} \cos \sqrt{\lambda}) \) which means that either \( C_2 = 0 \) or \( \sin \sqrt{\lambda} + \sqrt{\lambda} \cos \sqrt{\lambda} = 0 \). \( C_2 = 0 \) would imply \( C_3 = 0 \) which would lead to the trivial solution. Hence, we discard it. So, we must have \( \sin \sqrt{\lambda} + \sqrt{\lambda} \cos \sqrt{\lambda} = 0 \) that is \( \tan \sqrt{\lambda} = -\sqrt{\lambda} \). The equation \( \tan x = -x \) has no closed-form for its solution, we can still find its solution numerically. Since \( \tan x \) is periodic, we know that \( y = \tan x \) and \( y = -x \) will intersect at infinitely many points as figure shows 2.4.2.
CHAPTER 2. FOURIER’S METHOD: SEPARATION OF VARIABLES

Intersection of $y = \tan x$ and $y = -x$

Let $\lambda_n$ denote the $n^{th}$ eigenvalue for $n = 1, 2, 3, \ldots$ (remember that we only have a nontrivial solution when $\lambda > 0$). The corresponding eigenfunctions $X_n$ are

$$X_n (x) = C_2 \sin \sqrt{\lambda_n} x + C_3 \cos \sqrt{\lambda_n} x$$

and since $C_2 \sqrt{\lambda} = C_3$, we get

$$X_n (x) = C_2 \left( \sin \sqrt{\lambda_n} x + \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x \right)$$

Step 3: Finish Solving Using the IC

Since $T(t) = C_1 e^{-\lambda kt}$, using the superposition principle, the general solution is:

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n kt} \left( \sin \sqrt{\lambda_n} x + \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x \right)$$

where $A_n = C_1 C_2$. Since mixed boundary conditions are symmetric boundary condition, theorem 93 tells us that the eigenfunctions are orthogonal and that

$$A_n = \frac{(\phi (x), X_n (x))}{(X_n (x), X_n (x))} = \frac{\int_0^1 \phi (x) \left( \sin \sqrt{\lambda_n} x + \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x \right) dx}{\int_0^1 \left( \sin \sqrt{\lambda_n} x + \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x \right)^2 dx}$$

2.4.3 Problems

1. Using the results from the previous section, prove that the eigenvalues $\lambda$ in the example of this section must be real.

2. Using the results from the previous section, prove that the eigenvalues $\lambda$ in the example of this section must satisfy $\lambda \geq 0$.

3. Prove that the eigenvalue $\lambda$ in the example of this section which satisfies $\lambda = 0$ corresponds to the trivial solution.

4. Consider the BVP on $0 < x < 1$ given by

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) + X(0) = 0 \\ X(1) = 0 \end{cases}$$

(a) Explain why the eigenvalues must be real.

(b) Show that $\lambda = 0$ is an eigenvalue. Find the corresponding eigenfunction.

(c) Show there are no negative eigenvalues.

(d) Find an equation for all the positive eigenvalues.
(e) Show graphically that there are infinitely many positive eigenvalues.

(f) Compute up to 3 decimal places the numerical value of the first 3 positive eigenvalues. Find the corresponding eigenfunctions.

5. Consider
\[
\begin{aligned}
\text{PDE} & \quad \frac{\partial u}{\partial t} = u_{xx} \quad 0 < x < 1 \quad 0 < t < \infty \\
\text{BC} & \quad u_x(0, t) - u(0, t) = 0 \quad 0 < t < \infty \\
\text{IC} & \quad u_x(1, t) + u(1, t) = 0 \quad 0 < t < \infty \\
\end{aligned}
\]

(a) Give the physical interpretation of each line in this problem.

(b) Using the separation of variable method, write the PDE in two ODEs, one in \( T(t) \) and one in \( X(x) \).

(c) Find \( T(t) \).

(d) Write the eigenvalue problem in \( X \) (ODE and boundary conditions).

(e) Explain why the eigenvalues must be real.

(f) Explain why there are only positive eigenvalues.

(g) Find an equation of these positive eigenvalues. Find the corresponding eigenfunctions.

(h) Use the principle of superposition to find the general solution.

(i) Write the formula for the coefficients found in the previous question.
2.4.4 Answers to the Problems

1. Using the results from the previous section, prove that the eigenvalues $\lambda$ in the example of this section must be real.
   This is a proof, so there is no answer to give. I’ll give you a hint. Use theorem 96.

2. Using the results from the previous section, prove that the eigenvalues $\lambda$ in the example of this section must satisfy $\lambda \geq 0$.
   This is a proof, so there is no answer to give. I’ll give you a hint. Use theorem 97.

3. Prove that the eigenvalue $\lambda$ in the example of this section which satisfies $\lambda = 0$ corresponds to the trivial solution.
   Simply use the boundary conditions as we did in other examples.

4. Consider the BVP on $0 < x < 1$ given by
   \[
   \begin{align*}
   X'' + \lambda X &= 0 \\
   X'(0) + X(0) &= 0 \\
   X(1) &= 0
   \end{align*}
   \]
   (a) Explain why the eigenvalues must be real.
   Use theorem 96.

   (b) Show that $\lambda = 0$ is an eigenvalue. Find the corresponding eigenfunction.
   $X_0(x) = x - 1$, up to a constant multiple.

   (c) Show there are no negative eigenvalues.
   You can’t use theorem 97, you must check it.

   (d) Find an equation for all the positive eigenvalues.
   The eigenvalues are $\lambda_n$ where $\lambda_n$ is the $n^{th}$ solution of $\tan \sqrt{\lambda} = \sqrt{\lambda}$

   (e) Show graphically that there are infinitely many positive eigenvalues.
   Graph on the same graph $y = \tan x$ and $y = x$.

   (f) Compute up to 3 decimal places the numerical value of the first 3 positive eigenvalues. Find the corresponding eigenfunctions.
   Graph on the same graph $y = \tan x$ and $y = x$. Use the trace option of your favorite graphing tool and check that as you move the trace point around the point of intersection of the two graphs, the first 3 digits after the decimal point do not change. We find that
   \[
   \begin{align*}
   \sqrt{\lambda_1} &= 4.493 \\
   \sqrt{\lambda_2} &= 7.725 \\
   \sqrt{\lambda_3} &= 10.904
   \end{align*}
   \]
   The corresponding eigenfunctions, up to a constant multiple, are
   \[
   X_n(x) = \sin \sqrt{\lambda_n}x - \sqrt{\lambda_n} \cos \sqrt{\lambda_n}x
   \]
   for the given values of $\lambda_n$ that is for $n = 1, 2, 3$. 
5. Consider
\[
\begin{align*}
\text{PDE} & \quad u_t = u_{xx} \quad 0 < x < 1 \quad 0 < t < \infty \\
\text{BC} & \quad u_x (0, t) - u (0, t) = 0 \quad 0 < t < \infty \\
\text{IC} & \quad u (x, 0) = \phi (x) \quad 0 \leq x \leq 1
\end{align*}
\]
(a) Give the physical interpretation of each line in this problem.
One-dimensional heat equation for a thin rod of length 1 with initial temperature distribution equal to $\phi (x)$ with the surrounding temperature at both end equal to 0.
(b) Using the separation of variable method, write the PDE in two ODEs, one in $T (t)$ and one in $X (x)$.
\[
\begin{align*}
T' + \lambda k T &= 0 \\
X'' + \lambda X &= 0
\end{align*}
\]
(c) Find $T (t)$.
\[T (t) = C_1 e^{-\lambda kt}\]
(d) Write the eigenvalue problem in $X$ (ODE and boundary conditions).
\[
\begin{align*}
X'' + \lambda X &= 0 \\
X' (0) - X (0) &= 0 \\
X' (1) + X (1) &= 0
\end{align*}
\]
(e) Explain why the eigenvalues must be real.
Use theorem 96.
(f) Explain why there are only positive eigenvalues.
Use theorem 97 to see that there are no negative eigenvalues. Check that $\lambda = 0$ corresponds to the trivial solution.
(g) Find an equation of these positive eigenvalues. Find the corresponding eigenfunctions.
The eigenvalues are the positive solutions of $\tan \sqrt{x} = \frac{2\sqrt{x}}{x - 1}$. The corresponding eigenfunctions, up to a constant multiple, are $X_n (x) = \sin \sqrt{\lambda_n} x + \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x$ for $n = 1, 2, 3, \ldots$
(h) Use the principle of superposition to find the general solution. The general solution is
\[u (x, t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n k t} \left( \sin \sqrt{\lambda_n} x + \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x \right)\]
(i) Write the formula for the coefficients found in the previous question.
\[
A_n = \frac{\langle \phi (x), X_n (x) \rangle}{\langle X_n (x), X_n (x) \rangle} = \frac{\int_0^1 \phi (x) \left( \sin \sqrt{\lambda_n} x + \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x \right) dx}{\int_0^1 \left( \sin \sqrt{\lambda_n} x + \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x \right)^2 dx}
\]