1.4 Equivalence Relations

1.4.1 Definitions and Examples

Definition 73 (Relation) A binary relation or a relation on a set $S$ is a set $R$ of ordered pairs.

This is a very general definition. The ordered pairs simply list the elements which are related. If $(a, b) \in R$, we also write $aRb$ and it simply means that $a$ is in relation with $b$, whatever relation $R$ is. Here are a few examples.

Example 74 If $S = \mathbb{N}$, we define a relation $R$ by
\[ R = \{(a, b) \in \mathbb{N}^2 \mid a \text{ and } b \text{ are either both even or both odd}\} \]

Then, $(2, 6) \in R$ but $(2, 3) \notin R$.

Example 75 If $S = \mathbb{R}$, we define a relation $R$ by $R = \{(a, b) \in \mathbb{R}^2 \mid |a| = |b|\}$. Then, every real number $a$ is in relation with itself that is $(a, a) \in R$ since $|a| = |a|$. Also, $(-a, a) \in R$.

Example 76 If $S = \mathbb{Z}$, we define a relation $R$ by $R = \{(a, b) \in \mathbb{Z}^2 \mid a \leq b\}$. Here, our relation has its own special notation, $\leq$.

In its most general form, relations are not very interesting. However, relations with certain properties are more interesting. In this section, we will focus on equivalent relations. First, we list some of the properties a relation may have which are of interest.

Definition 77 Let $R$ be a relation defined on a set $S$ and let $a, b, c$ be elements of $S$.

1. $R$ is said to be reflexive if $\forall a \in S$, $(a, a) \in R$. In other words, every element in relation with itself.

2. $R$ is said to be symmetric if $\forall a, b \in S$, $(a, b) \in R \implies (b, a) \in R$.

3. $R$ is said to be antisymmetric if $\forall a, b \in S$, $(a, b) \in R$ and $(b, a) \in R \implies a = b$.

4. $R$ is said to be transitive if $\forall b, c \in S$, $(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R$.

We have the following definitions:

Definition 78 (Equivalence Relation) An equivalence relation on a set $S$ is a relation, that is a set $R$ of ordered pairs which has the following three properties:

1. reflexive.
2. symmetric.
3. transitive.

**Definition 79 (Order Relation)** An order relation on a set \( S \) is a relation, that is a set \( R \) of ordered pairs which has the following three properties:

1. reflexive.
2. antisymmetric.
3. transitive.

**Remark 80** If \( R \) is an equivalence relation, instead of writing \((a, b) \in R\) or \(aRb\), we often write \(a \sim b\). Using this notation, the reflexive property becomes \(a \sim a\), the symmetric property becomes \(a \sim b \implies b \sim a\), and the transitive property becomes \(a \sim b \text{ and } b \sim c \implies a \sim c\).

**Definition 81 (Equivalence Class)** If \( \sim \) is an equivalent relation on \( S \) and \( a \in S \), then the **equivalence class of \( a \) containing \( a \)**, denoted \([a]\), is defined to be:

\[ [a] = \{ x \in S \mid a \sim x \} \]

**Example 82** The relation \( R = \{(a, b) \in \mathbb{R}^2 \mid |a| = |b|\} \) is reflexive, symmetric, transitive. It is an equivalence relation.

**Example 83** The relation \( R = \{(a, b) \in \mathbb{Z}^2 \mid a \leq b\} \) is reflexive, antisymmetric and transitive. It is an order relation.

To illustrate how to prove a relation satisfies these properties, we give an example of a relation, and prove it is an equivalent relation.

**Example 84** The relation we are about to define is an important one. One we will use this semester. It has a special notation: \( \equiv \). If \( n \) is a positive integer and \( a \) and \( b \) are integers, we say that \( a \equiv b \) if \( a \mod n = b \mod n \) that is if \( n \mid (a - b) \).

- **reflexive**: Let \( a \in \mathbb{Z} \), we need to show that \( a \equiv a \) that is \( n \mid (a - a) \) which is true since any integer divides 0.

- **symmetric**: Let \( a \) and \( b \) be integers. We need to show that if \( a \equiv b \) then \( b \equiv a \).

\[
\begin{align*}
a &\equiv b \implies n \mid (a - b) \\
\implies a - b &= kn \text{ for some integer } n \\
\implies b - a &= (-k)n \\
\implies n \mid b - a \\
\implies b &\equiv a
\end{align*}
\]
• **transitive:** Let \( a, b \) and \( c \) be integers. We need to show that if \( a \equiv b \) and \( b \equiv c \) then \( a \equiv c \). If \( a \equiv b \) then \( a - b = k_1 n \). Similarly, \( b - c = k_2 n \). Therefore,

\[
a - c = a - b + b - c = (k_2 + k_2) n
\]

and therefore \( n \mid (a - c) \) or \( a \equiv c \).

• We can also find the equivalent classes of an element \( a \). By definition, we know that \( b \equiv a \) if \( n \mid b - a \) that is \( b - a = kn \) for some integer \( k \) or \( b = a + kn \). Hence \( [a] = \{a + kn\} \).

Equivalent classes play an important role as we will see in the next theorem.

**Definition 85 (Partition)** A partition of a set \( S \) is a collection of nonempty, disjoint subsets of \( S \) whose union is \( S \).

**Example 86** The sets \( \{2, 4, 6, 8, \ldots\} \) and \( \{1, 3, 5, 7, \ldots\} \) form a partition of \( \mathbb{N} \).

**Example 87** The sets \( \{0\} \), \( \{1, 2, 3, \ldots\} \), \( \{\ldots, -3, -2, -1\} \) form a partition of \( \mathbb{Z} \).

**Theorem 88** Let \( S \) be a set and \( R \) be an equivalence relation on \( S \). The equivalent classes of \( R \) form a partitions of \( S \). Conversely, for any partition \( P \) of \( S \), there is an equivalent class on \( S \) whose equivalent classes are the elements of \( P \).

**Proof.** We need to prove both directions.

• Let \( R \) be an equivalent relation on \( S \). We need to show that the equivalent classes of \( R \) form a partition of \( S \) that is they are not empty, disjoint and their union is \( S \). Let \( a \in S \). Clearly \( a \sim a \) from the reflexive property of \( \sim \). Thus, \( [a] \neq \emptyset \). Also, the union of all these classes is \( S \). Next, let \([a]\) and \([b]\) be distinct equivalence classes, we must show \([a] \cap [b] = \emptyset \). Suppose not, that is \( \exists c \in [a] \cap [b] \). We will show that it implies that both \([a] \subseteq [b]\) and \([b] \subseteq [a]\) that is \([a] = [b]\), which is a contradiction. We will only show \([a] \subseteq [b]\). The other inclusion is similar. Let \( x \in [a] \). Then, \( x \sim a \). Since \( c \in [a] \cap [b] \), \( a \sim c \) thus by transitivity, \( x \sim c \). But also \( c \sim b \), thus by transitivity, \( x \sim b \). It follows that \( x \in [b] \).

• Let \( P \) be a partition of \( S \) that is \( P \) is a collection of nonempty disjoint subsets of \( S \) whose union is \( S \). For \( a \) and \( b \) in \( S \), we define \( a \sim b \) if \( a \) and \( b \) belong to the same element of \( P \). It is easy to check that this is an equivalence relation. Obviously, \( a \) and \( a \) belong to the same subset of \( P \) so that \( a \sim a \). Also, if \( a \sim b \) then \( a \) and \( b \) belong to the same subset of \( P \). Thus \( b \) and \( a \) belong to the same subset of \( P \) hence \( b \sim a \). Finally, if \( a \sim b \) and \( b \sim c \) then \( a \) and \( b \) belong to the same subset of \( P \). So do \( b \) and \( c \). It follows that \( a \) and \( c \) belong to the same subset of \( P \) hence \( a \sim c \). So, \( \sim \) is reflexive, symmetric and transitive. It is an equivalence relation.
1.4.2 Exercises

1. Prove that if $a \in [b]$ then $[a] = [b]$.

2. What are the equivalent classes of the relation $\equiv$ defined by $a \equiv b$ if $a \mod n = b \mod n$ in the cases $n = 1, 4, 5$?

3. Do # 52, 53, 54 on page 24 of your book.