1.3 Induction

1.3.1 Definition and Examples

Proofs by induction are often used when one tries to prove a statement made about natural numbers or integers. Here are examples of statements where induction could be used.

- For every natural number \( n \), \( 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \)

- \((\cos x + i \sin x)^n = \cos nx + i \sin nx\). This is known as de Moivre’s theorem.

The principle of mathematical induction, states the following:

**Theorem 59 (First Principle of Mathematical Induction)** Let \( P(n) \) denote a statement about integers with the following properties:

1. The statement is true when \( n = 1 \) i.e. \( P(1) \) is true.
2. \( P(k+1) \) is true whenever \( P(k) \) is true for any integer \( k \geq 1 \).

Then, \( P(n) \) is true for every integer \( n \geq 1 \).

**Proof.** We use a proof by contradiction. Suppose the hypotheses of theorem 59 are true but the conclusion is false. That is, for some \( k \), \( P(k) \) is false. let \( A = \{k \in \mathbb{N} : P(k) \text{ is false}\} \). Then \( A \) is a subset of the positive integers and \( A \neq \emptyset \). So, it has a smallest element by the Well Ordering Principle, call it \( k_0 \). In particular, \( P(k_0 - 1) \) is true since \( k_0 \) is the smallest number for which \( P(k) \) is false. But by the hypotheses of the theorem, if \( P(k_0 - 1) \) is true, so should \( P(k_0) \). Which means \( k_0 \notin A \), which is a contradiction. So, \( P(n) \) must be true for all \( n \in \mathbb{N} \).

**Remark 60** The case \( n = 1 \) is called the base case.

**Remark 61** The principle of mathematical induction also holds if we replace 1 by any integer \( a \).

**Remark 62** When doing a proof by induction, it is important to write explicitly what the statement \( P(n) \) is so we know what we have to prove for a given \( n \). Before proving \( P(1) \), write clearly what \( P(1) \) says. Similarly, when we assume \( P(k) \) true and want to deduce \( P(k+1) \), write clearly what both \( P(k) \) and \( P(k+1) \) say so we know what we are assuming and what we need to prove.

We illustrate this principle with some examples which we state as theorems.

**Theorem 63** If \( n \) is a natural number, then \( 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \)

**Proof.** We do a proof by induction (though a nice direct proof also exists). Let \( P(n) \) denote the statement that \( 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \). We would
like to show that \( P(n) \) is true for all \( n \). \( P(1) \) states that \( 1 = \frac{1(1+1)}{2} \) which is true. This establishes that \( P(1) \) is true. Next, we assume that \( P(k) \) holds for some natural number \( k \). We wish to prove that \( P(k+1) \) also holds. We begin by writing what \( P(k) \) and \( P(k+1) \) represent so that we know what we are assuming and what we have to prove. \( P(k) \) says that \( 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \).

\[ P(k+1) \text{ says that } 1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}. \]

\[ 1 + 2 + 3 + \cdots + k + (k+1) = (1 + 2 + 3 + \cdots + k) + k + 1 \]
\[ = \frac{k(k+1)}{2} + k + 1 \text{ by assumption} \]
\[ = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \]
\[ = \frac{(k+1)(k+2)}{2} \]

So, we see that \( P(k+1) \) holds. Therefore, by induction, \( P(n) \) holds for all \( n \).

**Remark 64** Proving that \( P(1) \) is true is essential. Consider the statement \( n + 1 = n \) for all \( n \geq 0 \). This is obviously false. However, if we do not bother to check whether \( P(1) \) is true and we assume that \( P(k) \) is true then we can prove that \( P(k+1) \) is also true. \( P(k+1) \) says that \( n + 2 = n + 1 \).

\[ n + 2 = n + 1 \text{ by assumption since } n + 1 = n \]

Thus we would have proven that \( n + 1 = n \) that is \( 1 = 0 \).

**Theorem 65 (de Moivre Theorem)** \((\cos x + i \sin x)^n = \cos nx + i \sin nx\) where \( i \) is the complex number \( i = \sqrt{-1} \).

**Proof.** We do a proof by induction. Let \( P(n) \) be the statement that \( x \geq -1 \), and \( n \) is a natural number, then \((1 + x)^n \geq 1 + nx \).

- \( P(1) \) would be the statement \( \cos x + i \sin x = \cos x + i \sin x \), which is obviously true.

- Assume \( P(k) \) is true, that is \((\cos x + i \sin x)^k = \cos kx + i \sin kx\), we wish to prove that \( P(k+1) \) is also true, that is \((\cos x + i \sin x)^{k+1} = \cos (k+1)x + i \sin (k+1)x\).

We begin by reviewing some trigonometry

\[
\cos (a + b) = \cos a \cos b - \sin a \sin b \\
\sin (a + b) = \sin a \cos b + \cos a \sin b
\]
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\[
(\cos x + i \sin x)^{k+1} = (\cos x + i \sin x)^k (\cos x + i \sin x) \\
= (\cos kx + i \sin kx) (\cos x + i \sin x) \text{ by assumption} \\
= \cos kx \cos x + i \cos kx \sin x - \sin kx \sin x + i \sin kx \cos x \\
= \cos kx \cos x - \sin kx \sin x + i (\cos kx \sin x + \sin kx \cos x) \\
= \cos (k+1)x + i \sin (k+1)x
\]

Thus, \( P(k+1) \) holds. It follows by induction that \( P(n) \) holds for every \( n \).

Sometimes, it is not easy to deduce that \( P(k+1) \) is true knowing that \( P(k) \) is true, especially if we do not have a relationship between \( P(k) \) and \( P(k+1) \). In such cases, another form of mathematical induction can be used.

**Theorem 66 (Second Principle of Mathematical Induction)** Let \( P(n) \) denote a statement about integers with the following properties:

1. The statement is true when \( n = 1 \) i.e. \( P(1) \) is true.
2. \( P(k) \) is true whenever \( P(j) \) is true for all positive integers \( 1 \leq j < k \).

Then, \( P(n) \) is true for every integer \( n \geq 1 \).

**Example 67** Consider \( f : \mathbb{N} \to \mathbb{R} \) defined by \( f(1) = 0 \), \( f(2) = \frac{1}{3} \), and for \( n > 2 \) by \( f(n) = \frac{n-1}{n+1} f(n-2) \). By computing values of \( f(n) \) for \( n = 3, 4, 5, 6 \), give a conjecture as to what a direct formula for \( f \) might be. Prove your conjecture by induction.

- \( f(3) = \frac{2}{4} f(1) = \frac{1}{4} = 0 \)
- \( f(4) = \frac{3}{5} f(2) = \frac{3}{5} \frac{1}{3} = \frac{3}{15} = \frac{1}{5} \)
- \( f(5) = \frac{4}{6} f(3) = 0 \)
- \( f(6) = \frac{5}{7} f(4) = \frac{5}{7} \frac{1}{5} = \frac{1}{7} \)

- It seems that \( f(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{n+1} & \text{if } n \text{ is even} \end{cases} \). We need to prove this.

- Proof of the conjecture. We can see from the computations that the conjecture is true for \( n = 1, 2 \). Suppose that \( n > 2 \). Suppose our conjecture holds for all \( k < n \). We need to prove the conjecture also holds for \( n \). If
n is odd, then \( f(n) = \frac{n-1}{n+1} f(n-2) \). Since \( n \) is odd, so is \( n-2 \). Because \( n-2 < n \), the conjecture is true for \( n-2 \), so \( f(n-2) = 0 \) hence \( f(n) = 0 \). If \( n \) is even, then \( f(n) = \frac{n-1}{n+1} f(n-2) \). \( n-2 \) is also even and the conjecture holds for it. So, \( f(n-2) = 1 \) hence \( f(n) = 1 \). Therefore

\[
\begin{align*}
 f(n) &= \frac{n-1}{n+1} f(n-2) \\
&= \frac{n-1}{n+1} \frac{1}{n-1} \\
&= \frac{1}{n+1} 
\end{align*}
\]

The conjecture is proven.

Example 68 We use induction to prove the existence part of the fundamental theorem of arithmetic. Let \( P(n) \) be the statement that \( n \) is either prime or can be written as the product of primes. We wish to show \( P(n) \) is true for \( n > 1 \). Here, our base case is for \( n = 2 \). And 2 is prime, so \( P(2) \) is true. Now, let use assume that \( P(j) \) is true for all \( j \) such that \( 2 \leq j < k \). We must show \( P(k) \) is true, that is \( k \) is either prime or can be written as the product of primes. If \( k \) is prime, we are done. If \( k \) is not prime, then there exist integers \( a \) and \( b \) such that \( k = ab \) and \( a < k \) and \( b < k \). So, the induction hypothesis applies to \( a \) and \( b \) in other words, they can be written as a product of primes. Therefore, \( k = ab \) is also a product of primes. The uniqueness part is proven in the exercises.

1.3.2 Exercises

1. Prove by induction that if \( a_1, a_2, \ldots, a_n \) are all non-negative, then \( (1 + a_1)(1 + a_2) \ldots (1 + a_n) \geq 1 + a_1 + a_2 + \ldots + a_n \).

2. Prove by induction that \( \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1} \).

3. Use mathematical induction to show that the identities below are valid for any \( n \in \mathbb{N} \).

   (a) \( 1 + 3 + 5 + \ldots + (2n - 1) = n^2 \).

   (b) \( 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \).

   (c) \( 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[ \frac{1}{2} n(n+1) \right]^2 \).

   (d) \( 2 + 2^2 + 2^3 + \ldots + 2^n = 2(2^n - 1) \).

4. Use mathematical induction to establish the identities below for the given values of \( n \).
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(a) \(2^n > n\) for all \(n \in \mathbb{N}\).

(b) \(2^n > n^2\) for all \(n \in \mathbb{N}\) such that \(n \geq 5\).

(c) \(n! > 2^n\) for all \(n \in \mathbb{N}\) such that \(n \geq 4\).

(d) \(n! > 2^{n-1}\) for all \(n \in \mathbb{N}\) such that \(n > 1\).

5. In the questions below, \(f\) is a function with domain \(\mathbb{N}\). Use the given information to find a formula for \(f(n)\) then use mathematical induction to prove your formula is correct.

   (a) \(f(1) = \frac{1}{2}\), and for \(n > 1\), \(f(n) = (n - 1)f(n - 1) - \frac{1}{n + 1}\).

   (b) \(f(1) = 1\), \(f(2) = 4\), and for \(n > 2\), \(f(n) = 2f(n - 1) - f(n - 2) + 2\).

6. Let \(f : \mathbb{N} \rightarrow \mathbb{N}\) be defined recursively by:

   \[
   \begin{align*}
   f(1) &= 1 \\
   f(2) &= 2 \\
   f(n + 2) &= \frac{1}{2} [f(n + 1) + f(n)]
   \end{align*}
   \]

   Use mathematical induction to prove that \(1 \leq f(n) \leq 2\) for every \(n \in \mathbb{N}\).

7. Prove that if \(p\) is prime and \(p\) divides the product \(a_1a_2a_3\ldots a_n\) where \(a_i\) is an integer for \(i = 1, 2, \ldots, n\), then \(p \mid a_i\) for some \(i\). This is known as the Generalized Euclid’s lemma.

8. Use the previous problem to prove the uniqueness part of the fundamental theorem of arithmetic.