The Fundamental Homomorphism Theorem

Philippe B. Laval

KSU

Current Semester
In the previous chapter, we saw that every quotient group of \( G \) is a homomorphic image of \( G \).

In other words, if \( H \triangleleft G \) then \( G/H \) is a homomorphic image of \( G \).

In this chapter, we will prove the converse result, that is every homomorphic image of \( G \) is a quotient group of \( G \).

In other words, we will see that every homomorphic image of \( G \) is isomorphic to a quotient group of \( G \).
Preliminary Results

We begin with a result we will need for what follows.

**Theorem**

Let $G$ and $H$ be two groups and $f : G \rightarrow H$ be a homomorphism. Let $K = \ker f$. Then

$$f(a) = f(b) \iff Ka = Kb$$

**Remark:** Under the conditions of the theorem, we also have $f(a) = f(b) \iff aK = bK$. Why?
Sketch of a proof. Normally, we have to prove both directions. But the steps are reversible, so we can do both directions at the same time.

- Recall $Ka = Kb \iff ab^{-1} \in K$. How does this help?
Sketch of a proof. Normally, we have to prove both directions. But the steps are reversible, so we can do both directions at the same time.

- Recall $Ka = Kb \iff ab^{-1} \in K$. How does this help?
- Prove that $f(a) = f(b) \iff ab^{-1} \in K$ using the fact that $f$ is a homomorphism and the definition of $K$. 
We now state and prove the main result of this chapter.

**Theorem**

Suppose that $G$ is a group.

1. If $K \triangleleft G$, then the quotient group $G/K$ is a homomorphic image of $G$. In other words, the function $f : G \rightarrow G/K$ defined by $f(x) = Kx$ is a homomorphism of $G$ onto $G/K$ and $\ker f = K$.

2. If $H$ is a homomorphic image of $G$, then there exists a normal subgroup $K$ of $G$ such that $H$ is isomorphic to the quotient group $G/K$. More specifically, if $f : G \rightarrow H$ is a homomorphism of $G$ onto $H$, then $H$ is isomorphic to $G/K$ where $K = \ker f$. 

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Part 1 was proven with quotient groups.

Part 2: Suppose that $f : G \rightarrow H$ is a homomorphism of $G$ onto $H$ and let $K = \ker f$. We will show that $\phi : G/K \rightarrow H$ defined by $\phi(Kx) = f(x)$ for every $x \in G$ is an isomorphism. For this, we have four things to prove.

1. $\phi$ is well defined:
2. $\phi$ is injective:
3. $\phi$ is surjective:
4. $\phi$ preserves the operation:
Main Result: Sketch of a Proof

1. $\phi$ is well defined: Suppose that $x, y \in G$ and $Kx = Ky$. We need to show that $\phi(Kx) = \phi(Ky)$. Use the lemma and the definition of $\phi$. 

2. $\phi$ is injective: Suppose that $x, y \in G$ and $\phi(Kx) = \phi(Ky)$. We need to show that $Kx = Ky$. Use the definition of $\phi$, the fact $f$ is a homomorphism, and the result regarding cosets being equal.

3. $\phi$ is surjective: Let $z \in H$, we need to show that $z$ is the image of a coset of $K$ that is there exists $x \in G$: $(Kx) = z$. Use the fact that $f$ is onto.

4. $\phi$ preserves the operation: We need to show that for every $x, y \in G$, $(\phi(Kx) \phi(Ky)) = \phi(Kx \cdot Ky)$. Use coset multiplication, the definition of $\phi$, and the fact that $f$ is a homomorphism.
Main Result: Sketch of a Proof

1. \( \phi \) is well defined: Suppose that \( x, y \in G \) and \( Kx = Ky \). We need to show that \( \phi(Kx) = \phi(Ky) \).
   Use the lemma and the definition of \( \phi \).

2. \( \phi \) is injective: Suppose that \( x, y \in G \) and \( \phi(Kx) = \phi(Ky) \). We need to show that \( Kx = Ky \).
   Use the definition of \( \phi \), the fact \( f \) is a homomorphism, and the result regarding cosets being equal.

Philippe B. Laval (KSU)
The Fundamental Homomorphism Theorem
Current Semester 7 / 10
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3. $\phi$ is surjective: Let $z \in H$, we need to show that $z$ is the image of a coset of $K$ that is there exists $x \in G : \phi(Kx) = z$.
   Use the fact that $f$ is onto.
Main Result: Sketch of a Proof

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   Use the fact that \( f \) is onto.

4. \( \phi \) preserves the operation: We need to show that for every \( x, y \in G \),
   \( \phi((Kx)(Ky)) = \phi(Kx) \phi(Ky) \).
   Use coset multiplication, the definition of \( \phi \), and the fact that \( f \) is a homomorphism.
Consider the group of integers, $\mathbb{Z}$, with addition. As we know, $\mathbb{Z}$ is cyclic hence all its subgroups are cyclic. In addition, $\mathbb{Z}$ is Abelian, hence all its subgroups are normal subgroups.

Let us consider the subgroup

$$\langle 6 \rangle = \{ ..., -12, -6, 0, 6, 12, ... \}$$

The distinct cosets of this subgroup are

$$\langle 6 \rangle + 0 = \langle 6 \rangle = \{ ..., -12, -6, 0, 6, 12, ... \}$$
$$\langle 6 \rangle + 1 = \{ ..., -11, -5, 1, 7, 13, ... \}$$
$$\langle 6 \rangle + 2 = \{ ..., -10, -4, 2, 8, 14... \}$$
$$\langle 6 \rangle + 3 = \{ ..., -9, -3, 3, 9, 15, ... \}$$
$$\langle 6 \rangle + 4 = \{ ..., -8, -2, 4, 10, 16, ... \}$$
$$\langle 6 \rangle + 5 = \{ ..., -7, -1, 5, 11, 17, ... \}$$

and hence the elements of $\mathbb{Z}/\langle 6 \rangle$ are $\{ 0, 1, 2, 3, 4, 5 \}$ where $\bar{k} = \langle 6 \rangle + k$. 
By part 1 of the Fundamental Homomorphism Theorem, the function 
\( f : \mathbb{Z} \rightarrow \mathbb{Z}/\langle 6 \rangle \) defined by 
\( f(x) = \langle 6 \rangle + x \) is a homomorphism with 
\( \ker f = \langle 6 \rangle \).

To illustrate part 2 of the Fundamental Homomorphism Theorem, we 
ote that \( \mathbb{Z}_6 \) with addition modulo 6 is also a homomorphic image of 
\( \mathbb{Z} \). In particular, the function 
\( f : \mathbb{Z} \rightarrow \mathbb{Z}_6 \) defined by 
\( f(x) = x \mod 6 \) is a homomorphism of \( \mathbb{Z} \) onto \( \mathbb{Z}_6 \) and its kernel is 
\( \ker f = \langle 6 \rangle \). By part 2 of the theorem, \( \mathbb{Z}_6 \cong \mathbb{Z}/\langle 6 \rangle \).

The same argument can be made for \( \mathbb{Z}_n \) and \( \mathbb{Z}/\langle n \rangle \) for any integer \( n \).
1. Do the following problems at the end of chapter 16 of your book: A2, C1, C2, C3, F1, F2, I1, I2, I3, I4.