5.1 Definitions and Examples

5.1.1 Overview

In this chapter, we consider a function \( f : D \to \mathbb{R} \) where \( D \subseteq \mathbb{R} \). We also consider a real number \( a \) which is a limit point of \( D \). We will investigate the following limits: \( \lim_{x \to +\infty} f(x) \) and \( \lim_{x \to a} f(x) \). That is, we are studying how \( f \) behaves near a point \( a \) or near infinity. The questions we are trying to answer are:

1. As \( x \) is getting closer and closer to \( a \), how is \( y = f(x) \) behaving? Is it getting closer to a number as well? Is it getting arbitrary large (in absolute value)? Is it not following any pattern?

2. Same question if \( x \) is approaching \( +\infty \) or \( -\infty \).

In more general terms, we are asking the question: if \( x \) is following a certain pattern, is \( f(x) \) also following a pattern and if yes, which pattern?

In order to be able to evaluate \( \lim_{x \to a} f(x) \), \( f \) must be defined in a deleted neighborhood of \( a \) that is \( f \) must be defined in an interval of the form \((a - h, a + h)\) for some positive number \( h \), except maybe at \( x = a \).

When we say \( \lim_{x \to a} f(x) = L \), we mean that \( f(x) \) can be made as close as we want from \( L \) simply by taking \( x \) close enough to \( a \). Or, in terms of neighborhoods, we have the following general definition for a limit.

**Definition 5.1.1** We say that \( \lim_{x \to a} f(x) = L \) or that \( f(x) \to L \) as \( x \to a \) if for every neighborhood \( V \) of \( L \), one can find a deleted neighborhood \( U \) of \( a \) such that \( x \in U \implies f(x) \in V \).

This definition can be adapted to limits at a finite point or at infinity as well as when the limit is finite or infinite. There are three possibilities for \( x \), we
can have $x \to a$, $x \to -\infty$ and $x \to \infty$. For each case, we can have $f(x) \to L$, $f(x) \to -\infty$ and $f(x) \to \infty$. Hence, we have a total of nine definitions. They can all be derived from the above definition simply by remembering that a neighborhood of a finite point $a$ is an interval of the form $(a - \delta, a + \delta)$ and a neighborhood of infinity is an interval of the form $(w, \infty)$ for some $w \in \mathbb{R}$.

5.1.2 Limit at a finite point

Definition 5.1.2 We say that \( \lim_{x \to a} f(x) = L \) or that \( f(x) \to L \) as \( x \to a \) if
\[
\forall \epsilon > 0, \exists \delta > 0 : 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.
\]

\( |x - a| \) represents how far \( x \) is from \( a \). The above statement says that \( f(x) \) can be made arbitrarily close to \( L \) simply by taking \( x \) close enough to \( a \).

Example 5.1.3 Prove that \( \lim_{x \to 2} (x + 5) = 7 \).

Given \( \epsilon > 0 \), we must prove that there exists \( \delta > 0 \) such that \( 0 < |x - 2| < \delta \implies |x + 5 - 7| < \epsilon \). Let \( \epsilon > 0 \) be given.

\[
|x + 5 - 7| < \epsilon \iff |x - 2| < \epsilon
\]

Thus we see that \( \delta = \epsilon \) will work. Indeed, given \( \epsilon > 0 \), \( 0 < |x - 2| < \delta \implies |x + 5 - 7| < \epsilon \).

Remark 5.1.4 Of course, this was a very easy example to illustrate how this kind of problem is addressed. In general, it will take more work to find \( \delta \) given \( \epsilon > 0 \). Many of the techniques used for sequences will also be used here. Some will be illustrated below when we look at more challenging examples.

Remark 5.1.5 Saying that \( 0 < |x - a| < \delta \) is the same as saying that \( x \in (a - \delta, a + \delta) \) and \( x \neq a \). Similarly, saying that \( |f(x) - L| < \epsilon \) is the same as saying that \( f(x) \in (L - \epsilon, L + \epsilon) \).

Remark 5.1.6 In order to understand how to adapt this definition to cases involving \( \infty \) such as in the case when \( L = \infty \) or also when \( x \to \infty \), it is important to understand the notion of neighborhood of \( \infty \). A neighborhood of \( \infty \) is an interval of the form \((w, \infty)\). Similarly, an neighborhood of \( -\infty \) is an interval of the form \((-\infty, w)\).

Definition 5.1.7 We say that \( \lim_{x \to a} f(x) = \infty \) or that \( f(x) \to \infty \) as \( x \to a \) if
\[
\forall M > 0, \exists \delta > 0 : 0 < |x - a| < \delta \implies f(x) > M.
\]

Example 5.1.8 Prove that \( \lim_{x \to 1} \frac{1}{(x - 1)^2} = \infty \).

Given \( M > 0 \), we must prove that there exists \( \delta > 0 \) such that \( 0 < |x - 1| <
\[\delta \implies \frac{1}{(x-1)^2} > M.\]

\[\frac{1}{(x-1)^2} > M \iff (x-1)^2 < \frac{1}{M} \]

\[\iff -\frac{1}{\sqrt{M}} < x-1 < \frac{1}{\sqrt{M}} \]

\[\iff |x-1| < \frac{1}{\sqrt{M}}.\]

Thus, given \(M > 0\), \(\delta = \frac{1}{\sqrt{M}}\) will work.

**Definition 5.1.9** We say that \(\lim_{x \to a} f(x) = -\infty\) or that \(f(x) \to -\infty\) as \(x \to a\) if \(\forall M < 0\), \(\exists \delta > 0\) : \(0 < |x-a| < \delta \implies f(x) < M\).

**Remark 5.1.10** If we find a \(\delta\) which works, then every \(\delta' < \delta\) will also work. Therefore, it is always possible to impose certain conditions on \(\delta\) such as saying that we are looking for \(\delta\) less than a certain number \(h\), thus restricting our search to an interval of the form \((a-h, a+h)\). In this interval, if we call \(\delta'\) the value we found, then \(\delta = \min(h, \delta')\).

**Remark 5.1.11** In the last two definition, the vertical line \(x = a\) is a vertical asymptote for the graph of \(y = f(x)\).

**Remark 5.1.12** In the definition of a limit, it is implied that \(a\) is a limit point of \(D(f)\) that is for every \(\delta > 0\) the interval \((a-\delta, a+\delta)\) contains points of \(D(f)\) other than \(a\). If this is not the case, then for \(\delta\) small enough, there may not exist any \(x\) satisfying \(0 < |x-a| < \delta\). In this case, the concept of a limit has no meaning.

**Remark 5.1.13** In the definition of a limit, \(a\) does not have to be in the domain of \(f\). It only needs to be a limit point of \(D(f)\).

**Remark 5.1.14** In the definition of a limit, \(\delta\) depends obviously on \(\epsilon\). It may also depend on the point \(a\) as illustrated by example 5.1.38.

**Remark 5.1.15** When we say that the limit of a function exists, we mean that it exists and is finite. When the limit is infinite, it does not exist in the sense that it is not a number. However, we know what the function is doing, it is approaching \(\pm \infty\).

**Remark 5.1.16** There are several situations under which a limit will fail to exist.

1. The function may oscillate boundedly like in \(f(x) = \sin \frac{1}{x}\) as \(x \to 0\).
2. The function may oscillate unboundedly like \(x \sin x\) as \(x \to \infty\).
3. The function may grow without bounds like \(\frac{1}{x^2}\) as \(x \to 0\).
4. There may be a "break" in the graph.
Limit at infinity

In order to be able to evaluate \( \lim_{x \to \infty} f(x) \), \( f \) must be defined for large \( x \). In other words, we must have \( D(f) \cap (w, \infty) \neq \emptyset \) for every \( w \in \mathbb{R} \). In the case we want to evaluate \( \lim_{x \to \infty} f(x) \), then we must have \( D(f) \cap (-\infty, w) \neq \emptyset \) for every \( w \in \mathbb{R} \). We then have the following definitions (some of the definitions will be accompanied with easy examples to illustrate the concept being defined):

**Definition 5.1.17** We say that \( \lim_{x \to \infty} f(x) = L \) or that \( f(x) \to L \) as \( x \to \infty \) if \( \forall \epsilon > 0, \exists w > 0 : x \in (w, \infty) \cap D(f) \implies |f(x) - L| < \epsilon \)

\( |f(x) - L| \) represents the distance between \( f(x) \) and \( L \). The above statement simply says that \( f(x) \) can be made as close as one wants to \( L \), simply by taking \( x \) large enough. Graphically, this simply says that the line \( y = L \) is a horizontal asymptote for the graph of \( y = f(x) \).

To prove that a number \( f(x) \) approaches \( L \) as \( x \to \infty \), given \( \epsilon > 0 \), one has to prove that \( w > 0 \) can be found so that \( x \in (w, \infty) \cap D(f) \implies |f(x) - L| < \epsilon \). The approach is very similar to the one used for sequences. Many of the techniques used when finding the limit of a sequence will also be used here.

**Example 5.1.18** Prove that \( \lim_{x \to \infty} \frac{1}{x} = 0 \).

Let \( \epsilon > 0 \) be given. We want to find \( w > 0 \) so that \( x \in (w, \infty) \cap D(f) \implies \left| \frac{1}{x} - 0 \right| < \epsilon \). As usual, we begin with the inequality we are trying to prove.

\[
\left| \frac{1}{x} - 0 \right| < \epsilon \iff \frac{1}{|x|} < \epsilon
\]

Since we are considering the limit as \( x \to \infty \), we can restrict ourselves to positive values of \( x \). Thus, the above inequality can be replaced with \( \frac{1}{x} < \epsilon \) which is equivalent to \( x > \frac{1}{\epsilon} \). Thus we see that given \( \epsilon > 0 \), \( w = \frac{1}{\epsilon} \) will work.

Thus, given \( \epsilon > 0 \), we have \( x \in \left( \frac{1}{\epsilon}, \infty \right) \cap D(f) \implies \left| \frac{1}{x} - 0 \right| < \epsilon \).

**Definition 5.1.19** We say that \( \lim_{x \to \infty} f(x) = \infty \) or that \( f(x) \to \infty \) as \( x \to \infty \) if \( \forall M > 0, \exists w > 0 : x \in (w, \infty) \cap D(f) \implies f(x) > M \).

The above definition says that \( f(x) \) can be made arbitrarily large, simply by taking \( x \) large enough.

**Example 5.1.20** Prove that \( \lim_{x \to \infty} x^2 = \infty \).

Given \( M > 0 \), we must prove that there exists \( w > 0 \) such that \( x \in (w, \infty) \cap D(x^2) \implies x^2 > M \). Since we are considering the limit as \( x \to \infty \), we can restrict ourselves to \( x > 0 \). In this case

\[
x^2 > M \iff x > \sqrt{M}
\]
Thus, we see that given $M > 0$, \( w = \sqrt{M} \) will work in other words $x \in \left( \sqrt{M}, \infty \right) \iff x^2 > M$.

The remaining definitions are given below.

**Definition 5.1.21** We say that $\lim_{x \to \infty} f(x) = -\infty$ or that $f(x) \to -\infty$ as $x \to \infty$ if $\forall M < 0$, $\exists w > 0 : x \in (w, \infty) \cap D(f) \implies f(x) < M$.

**Definition 5.1.22** We say that $\lim_{x \to -\infty} f(x) = L$ or that $f(x) \to L$ as $x \to -\infty$ if $\forall \epsilon > 0$, $\exists w < 0 : x \in (-\infty, w) \cap D(f) \implies |f(x) - L| < \epsilon$.

**Definition 5.1.23** We say that $\lim_{x \to -\infty} f(x) = \infty$ or that $f(x) \to \infty$ as $x \to -\infty$ if $\forall M > 0$, $\exists w < 0 : x \in (-\infty, w) \cap D(f) \implies f(x) > M$.

**Definition 5.1.24** We say that $\lim_{x \to \infty} f(x) = -\infty$ or that $f(x) \to -\infty$ as $x \to -\infty$ if $\forall M < 0$, $\exists w > 0 : x \in (-\infty, w) \cap D(f) \implies f(x) < M$.

**Remark 5.1.25** In the above definitions, $x \in (w, \infty)$ can be replaced by $x > w$ and $x \in (-\infty, w)$ can be replaced by $x < w$.

**One-sided Limits**

When we say $x \to a$, we realize that $x$ can approach $a$ from two sides. If $x$ approaches $a$ from the right, that is if $x$ approaches $a$ and is greater than $a$, we write $x \to a^+$. Similarly, if $x$ approaches $a$ from the left, that is if $x$ approaches $a$ and is less than $a$, then we write $x \to a^-$.

We can rewrite the above definition for one sided limits with little modifications. We do it for a few of them.

**Definition 5.1.26** We say that $\lim_{x \to a^+} f(x) = L$ or that $f(x) \to L$ as $x \to a^+$ if $\forall \epsilon > 0$, $\exists \delta > 0 : 0 < x - a < \delta \implies |f(x) - L| < \epsilon$.

**Definition 5.1.27** We say that $\lim_{x \to a^-} f(x) = L$ or that $f(x) \to L$ as $x \to a^-$ if $\forall \epsilon > 0$, $\exists \delta > 0 : 0 < a - x < \delta \implies |f(x) - L| < \epsilon$.

**Example 5.1.28** Prove that $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

Given $M > 0$, we need to prove that there exists $\delta > 0$ such that $0 < x < 0 < \delta \implies \frac{1}{x} > M$.

\[
\frac{1}{x} > M \iff x < \frac{1}{M}
\]

Thus, given $M > 0$, we see that $\delta = \frac{1}{M}$ will work in other words, we will have $0 < x - 0 < \frac{1}{M} \implies \frac{1}{x} > M$. 
Theorem 5.1.29  The following two conditions are equivalent

1. \( \lim_{x \to a} f(x) = L \)
2. \( \lim_{x \to a^+} f(x) = L \) and \( \lim_{x \to a^-} f(x) = L \)

Proof. See problems. \( \blacksquare \)

Remark 5.1.30  One way to prove that \( \lim_{x \to a} f(x) \) does not exist is to prove that the two one-sided limits are not the same or that at least one of them does not exist.

Remark 5.1.31  One sided limits are often used when the definition or behavior of \( f \) changes around the point \( a \) at which the limit is being computed. This can happen with piecewise functions when we compute their limit at one of the breaking points.

We now look at several examples illustrating various techniques used when computing limits.

5.1.3  Computing Limits Using the definitions: Examples

Example 5.1.32  Show that \( \lim_{x \to 3} (4x - 5) = 7 \)

We need to show that given \( \epsilon > 0 \), one can find \( \delta \) such that \( 0 < |x - 3| < \delta \implies |(4x - 5) - 7| < \epsilon \). The strategy is to start with \( |(4x - 5) - 7| < \epsilon \) and change the absolute value to get an inequality with \( jx - 3j \).

\[
|4x - 12| < \epsilon
\]

\[
\iff 4|x - 3| < \epsilon
\]

\[
\iff |x - 3| < \frac{\epsilon}{4}
\]

So, given \( \epsilon > 0 \), \( \delta = \frac{\epsilon}{4} \) will work.

Example 5.1.33  Show that \( \lim_{x \to 2} (x^2 + 2) = 6 \)

We need to show that given \( \epsilon > 0 \), one can find \( \delta \) such that \( 0 < |x - 2| < \delta \implies |(x^2 + 2) - 6| < \epsilon \).

\[
|(x^2 + 2) - 6| < \epsilon \iff |x^2 - 4| < \epsilon
\]

\[
\iff |x - 2||x + 2| < \epsilon
\]

Since \( x^2 + 2 \) is defined for all real numbers, we can assume it is at least defined in \((0, 4)\) (i.e. \( (2 - h, 2 + h) \) with \( h = 2 \)). In this interval, \( |x + 2| < 6 \), therefore, \( |x - 2||x + 2| < 6|x - 2| \). So if we make \( 6|x - 2| < \epsilon \), the result will follow. This happens when \( |x - 2| < \frac{\epsilon}{6} \). So, given \( \epsilon > 0 \), \( \delta = \min \left( \frac{\epsilon}{6}, 2 \right) \) will work.
Remark 5.1.34 In the previous example, the choice of the interval \((0, 4)\) is not magic. We could have chosen another interval. We want to use an interval centered at the point where we are computing the limit. Some readers may think we cheated by only looking at values of \(x\) in some intervals. Remember what we are trying to achieve. Given an \(\varepsilon > 0\), we are finding \(\delta > 0\) with certain properties. As long as we find a \(\delta\), we have achieved what we had to achieve. By picking an interval, we simply acknowledge the fact that it is too difficult to look for just any \(\delta\), so we restrict our search to a smaller interval.

Example 5.1.35 Show that \(\lim_{x \to 4} \sqrt{x} = 2\).
We need to show that given \(\varepsilon > 0\), one can find \(\delta > 0\) such that \(|\sqrt{x} - 2| < \varepsilon\). As before, we start with what we want, and manipulate it until we get \(|\sqrt{x} - 2|\) involved.

\[
|\sqrt{x} - 2| < \varepsilon \iff \left| \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{\sqrt{x} + 2} \right| < \varepsilon
\]

\[
\iff \frac{|x - 4|}{\sqrt{x} + 2} < \varepsilon
\]

Since we are computing the limit as \(x \to 4\), we are looking at values of \(x\) in an interval of the form \((4 - h, 4 + h)\). If we restrict ourselves to \(h = 2\), then we are looking at values of \(x\) in \((2, 6)\). There, \(\sqrt{x} + 2 > \sqrt{2} + 2 > 2\). Hence, \(\frac{|x - 4|}{\sqrt{x} + 2} < \frac{|x - 4|}{2} < \varepsilon\). So, if we make \(\frac{|x - 4|}{2} < \varepsilon\) that is \(|x - 4| < 2\varepsilon\), the result will follow. So, given \(\varepsilon > 0\), \(\delta = \min(2\varepsilon, 2)\) will work.

Our next example illustrates how to work with piecewise functions.

Example 5.1.36 Let \(f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 < x \leq 2 \\ 8 - x & \text{if } x > 2 \end{cases}\)

1. Prove that \(\lim_{x \to 0} f(x) = 0\).
Since the definition of \(f\) changes at 0, we will need to consider one-sided limits. We can prove that \(\lim_{x \to 0^+} f(x) = 0\) by proving that \(\lim_{x \to 0^+} f(x) = 0\) and \(\lim_{x \to 0^-} f(x) = 0\).

- \(\lim_{x \to 0^-} f(x) = 0\).
  Let \(\varepsilon > 0\) be given. We want to prove there exists \(\delta > 0\) such that \(0 < 0 - x < \delta \iff |f(x) - 0| < \varepsilon\). If \(x \to 0^-\) then \(x < 0\). In this
case, \( f(x) = x \). Thus, we have:

\[
|f(x) - 0| < \epsilon \iff |x - 0| < \epsilon
\]

\[
\iff |x| < \epsilon
\]

\[
\iff -\epsilon < x < \epsilon
\]

\[
\iff -\epsilon < x < 0 \text{ (since } x < 0)\]

Thus given \( \epsilon > 0 \), \( \delta = \epsilon \) will work.

- \( \lim_{x \to 0^+} f(x) = 0 \).
  
  Let \( \epsilon > 0 \) be given. We want to prove there exists \( \delta > 0 \) such that
  
  \[ 0 < x - 0 < \delta \iff |f(x) - 0| < \epsilon. \]

  If \( x \to 0^+ \) then \( x > 0 \). In this case, \( f(x) = x^2 \). Thus, we have:

  \[
  |f(x) - 0| < \epsilon \iff |x^2 - 0| < \epsilon
  \]

  \[
  \iff x^2 < \epsilon
  \]

  \[
  \iff |x| < \sqrt{\epsilon}
  \]

  \[
  \iff 0 < x < \sqrt{\epsilon} \text{ (since } x > 0)\]

  Thus, given \( \epsilon > 0 \), we see that \( \delta = \sqrt{\epsilon} \) will work.

2. Prove that \( \lim_{x \to 2} f(x) \) does not exist.

As for the previous question, we need to consider one-sided limits. We will prove that \( \lim_{x \to 2} f(x) \) does not exist by proving the one-sided limits are different. more specifically, we prove that \( \lim_{x \to 2^+} f(x) = 4 \) and \( \lim_{x \to 2^-} f(x) = 6 \).

- \( \lim_{x \to 2^-} f(x) = 4 \).
  
  Let \( \epsilon > 0 \) be given. We want to prove there exists \( \delta > 0 \) such that
  
  \[ 0 < 2 - x < \delta \iff |f(x) - 4| < \epsilon. \]

  If \( x \to 2^- \) then \( x < 2 \). In this case, \( f(x) = x^2 \). Thus, we have:

  \[
  |f(x) - 4| < \epsilon \iff |x^2 - 4| < \epsilon
  \]

  \[
  \iff |x - 2||x + 2| < \epsilon
  \]

  Here, we use a technique similar to one used in an example above. Since we are computing the limit of \( f(x) \) as \( x \to 2^- \), \( f \) has to be defined in an interval of the form \((2 - h, 2)\) for some \( h \). We can restrict ourselves to \((0, 2)\) (this is the case when \( h = 2 \)). Indeed, \( f \) is defined in this interval. Also, in this interval, \(|x + 2| < 4\) and therefore

  \[
  |f(x) - 4| < \epsilon \iff |x - 2| < \frac{\epsilon}{4}
  \]

  Thus, we see that given \( \epsilon > 0 \), \( \delta = \min \left( \frac{\epsilon}{4}, 2 \right) \) will work.
• \( \lim_{x \to 2^+} f(x) = 6. \)

Let \( \epsilon > 0 \) be given. We want to prove there exists \( \delta > 0 \) such that \( 0 < x - 2 < \delta \iff |f(x) - 6| < \epsilon. \) If \( x \to 2^+ \) then \( x > 2. \) In this case, \( f(x) = 8 - x. \) Thus, we have:

\[
|f(x) - 6| < \epsilon \iff |8 - x - 6| < \epsilon \\
\iff |-x + 2| < \epsilon \\
\iff |x - 2| < \epsilon
\]

We see that in this case given \( \epsilon > 0, \delta = \epsilon \) will work.

Example 5.1.37 Find \( \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}. \)

First, we note that if \( x \neq 4, \) we have:

\[
\frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{x} - 2}{x - 4} \frac{\sqrt{x} + 2}{\sqrt{x} + 2} \\
= \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{(x - 4)(\sqrt{x} + 2)} \\
= \frac{(\sqrt{x})^2 - 4}{(x - 4)(\sqrt{x} + 2)} \\
= \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} \\
= \frac{1}{\sqrt{x} + 2}
\]

So that when \( x \to 4, \) it seems reasonable to think that \( \frac{\sqrt{x} - 2}{x - 4} = \frac{1}{\sqrt{x} + 2} \to \frac{1}{4}. \)

We prove it. Given \( \epsilon > 0, \) we need to find \( \delta > 0 \) such that \( 0 < |x - 4| < \delta \implies \)
\[
\left| \frac{\sqrt{x} - 2}{x - 4} - \frac{1}{4} \right| < \epsilon.
\]

\[
\left| \frac{\sqrt{x} - 2}{x - 4} - \frac{1}{4} \right| < \epsilon \iff \left| \frac{1}{\sqrt{x} + 2} - \frac{1}{4} \right| < \epsilon
\]
\[
\iff \left| \frac{4 - \sqrt{x} - 2}{4(\sqrt{x} + 2)} \right| < \epsilon
\]
\[
\iff \left| \frac{2 - \sqrt{x}}{4(\sqrt{x} + 2)} \right| < \epsilon
\]
\[
\iff \left| \frac{2 - \sqrt{x}}{4(\sqrt{x} + 2)} \right| < \epsilon
\]
\[
\iff \left| \frac{(2 - \sqrt{x})(2 + \sqrt{x})}{4(\sqrt{x} + 2)^2} \right| < \epsilon
\]
\[
\iff \left| \frac{(4 - x)}{4(\sqrt{x} + 2)^2} \right| < \epsilon
\]
\[
\iff \left| \frac{4 - x}{4(\sqrt{x} + 2)^2} \right| < \epsilon
\]

Since

\[
\frac{|4 - x|}{4(\sqrt{x} + 2)^2} = \frac{|x - 4|}{4(\sqrt{x} + 2)^2} < \frac{|x - 4|}{4x}
\]

Since \( x \to 4 \), we can restrict ourselves to the interval \((3, 5)\) that is an interval of the form \((4 - h, 4 + h)\) with \( h = 1 \). In this interval \( 12 < 4x < 20 \). Thus, in this interval

\[
\frac{|x - 4|}{4x} < \frac{|x - 4|}{12}
\]

So, we make \( \frac{|x - 4|}{12} < \epsilon \) the result will follow. This will happen if and only if \( |x - 4| < 12\epsilon \). Thus, given \( \epsilon > 0 \), \( \delta = \min(1, 12\epsilon) \) will work. It follows that

\[
\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{1}{4}
\]

The next example illustrates the fact that \( \delta \) depends not only on \( \epsilon \) but also on the point at which the limit is being found.

**Example 5.1.38** Prove that \( \lim_{x \to a} \frac{1}{x} = \frac{1}{a} \) for any \( a \in (0, \infty) \).

Let \( \epsilon > 0 \) be given. We want to find \( \delta > 0 \) such that \( 0 < |x - a| < \delta \implies \)
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\[
\left| \frac{1}{x} - \frac{1}{a} \right| < \varepsilon. \quad \text{We begin with}
\]

\[
\left| \frac{1}{x} - \frac{1}{a} \right| < \varepsilon \iff \left| \frac{a-x}{ax} \right| < \varepsilon
\]

\[
\iff \frac{|x-a|}{ax} < \varepsilon
\]

When we are looking at values of \( x \) satisfying \(|x-a| < \delta\) the intent is that \( \delta \) be small in other words that be \( x \) is close to \( a \). We may then assume that \(|x-a| < \frac{a}{2}\) in which case \( x > \frac{a}{2} \). Therefore \( \frac{|x-a|}{ax} < \frac{2|x-a|}{a^2} \). So, if we make \( \frac{2|x-a|}{a^2} < \varepsilon \) we will have what we want. This will happen when \( |x-a| < \frac{a^2 \varepsilon}{2} \).

Thus, we see that if \( \delta = \min \left( \frac{a}{2}, \frac{a^2 \varepsilon}{2} \right) \) then \( 0 < |x-a| < \delta \implies \left| \frac{1}{x} - \frac{1}{a} \right| < \varepsilon \).

In particular, we see that \( \delta \) depends on both \( \varepsilon \) and \( a \). Of course, one might argue that we could have done better and found a \( \delta \) which did not depend on \( a \). We will prove by contradiction that \( \delta \) must depend on \( a \). Suppose that it did not, that is for a given \( \varepsilon > 0 \), the choice of \( \delta \) does not depend on \( a \). In particular, this would work for \( a = 1 \). So, for \( \varepsilon = 1 \), one can find \( \delta > 0 \) such that \( 0 < |x-a| < \delta \implies \left| \frac{1}{x} - \frac{1}{a} \right| < 1 \). If it works for this \( \delta \), it must also work for any smaller \( \delta \). We may then assume that \( 0 < \delta < \frac{1}{2} \). Since the choice of \( \delta \) is supposed to be independent of \( a \), it should work for \( a = \frac{\delta}{2} \). If \( x = \delta \), we have

\[
0 < |x-a| = \left| \delta - \frac{\delta}{2} \right| = \left| \frac{\delta}{2} \right| < \delta
\]

Therefore, we should have \( \left| \frac{1}{x} - \frac{1}{a} \right| < 1 \). But

\[
\left| \frac{1}{x} - \frac{1}{a} \right| = \left| \frac{1}{\delta} - \frac{2}{\delta} \right|
\]

\[
= \frac{1}{\delta} > 1
\]

which is a contradiction.

Example 5.1.39 Let \( f : \mathbb{R} \to \mathbb{R} \) defined by

\[
f(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q} \\
0 & \text{if } x \notin \mathbb{Q}
\end{cases}
\]

Prove that \( \lim_{x \to a} f(x) \) does not exist for any \( a \in \mathbb{R} \).

Fix \( a \in \mathbb{R} \). We show that if \( L \in \mathbb{R} \) then \( L \) cannot be the limit of \( f(x) \) as \( x \to a \).
by showing that there exists $\epsilon > 0$ for which no $\delta$ will work that is no matter which $\delta$ we pick, there exists an $x$ satisfying $0 < |x - a| < \delta$ yet $|f(x) - L| \geq \epsilon$. Let $\epsilon = \max \{|L - 1|, |L|\}$. Either $\epsilon = |L - 1|$ or $\epsilon = |L|$.

**case 1:** $\epsilon = |L - 1|$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, one can find $x \in \mathbb{Q}$ such that for any $\delta > 0$ we have

$$0 < |x - a| < \delta$$

For such $x$, $|f(x) - L| = |1 - L| = \epsilon$.

**case 2:** $\epsilon = |L|$. Since between any two real numbers there exists an irrational number, for any $\delta > 0$ one can find an irrational number $x$ such that

$$0 < |x - a| < \delta$$

For such $x$, $|f(x) - L| = |L| = \epsilon$.

**Conclusion:** We have proven that no matter which $\delta$ we pick, one can find $x$ satisfying $0 < |x - a| < \delta$ yet $|f(x) - L| \geq \epsilon$. This shows $L$ cannot be a limit of $f$ at $a$. Since $L$ was arbitrary, $\lim_{x \to a} f(x)$ does not exist.

### 5.1.4 Exercises

1. Use the definition of the limit of a function to show that $\lim_{x \to \sqrt{2}} \sqrt{x} = \sqrt{2}$.

2. Use the definition of the limit of a function to show that $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$.

3. Discuss the one-sided limits of $f(x) = e^{\frac{1}{x}}$ at $x = 0$.

4. Write a careful proof of the results stated below.

   (a) $\lim_{x \to 1} (x^3 - 3x) = 2$

   (b) $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$

   (c) $\lim_{x \to 2} \frac{x^3 - 8}{x^2 + x - 6} = \frac{12}{5}$

5. Let $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$

   Prove the following:

   (a) $\lim_{x \to 1} f(x) = 1$

   (b) $\lim_{x \to 2} f(x)$ does not exist.

6. Prove theorem [5.1.29]
5.1. DEFINITIONS AND EXAMPLES

5.1.5 Hints for the Exercises

1. Use the definition of the limit of a function to show that \[ \lim_{x \to 2} \sqrt{x} = \sqrt{2} \].
   Hint: As the problem says, use the definition. You will also need to multiply by the conjugate.

2. Use the definition of the limit of a function to show that \[ \lim_{x \to -2} \frac{1}{x} = \frac{1}{2} \].
   Hint: As in some of the examples, you will need to restrict your search to some interval around \(-2\), the number \(x\) is approaching.

3. Discuss the one-sided limits of \( f(x) = e^{\frac{1}{x}} \) at \( x = 0 \).
   Hint: Study carefully the behavior of \( \frac{1}{x} \) as \( x \to 0 \).

4. Write a careful proof of the results stated below.
   (a) \[ \lim_{x \to 1} (x^3 - 3x) = 2 \]
      Hint: As in some of the examples, you will need to restrict your search to some interval around 1, the number \(x\) is approaching.
   (b) \[ \lim_{x \to 3} \frac{1}{x} = \frac{1}{3} \]
      Hint: Similar hint as above.
   (c) \[ \lim_{x \to 2} \frac{x^3 - 8}{x^2 + x - 6} = \frac{12}{5} \]
      Hint: Similar hint as above.

5. Let \( f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases} \)
   Prove the following:
   (a) \[ \lim_{x \to 1} f(x) = 1 \]
      Hint: Using the definition of limits, write what you have to prove, and use techniques similar to the previous problems and the fact that both rationals and irrationals are dense in \( \mathbb{R} \).
   (b) \[ \lim_{x \to 2} f(x) \text{ does not exist.} \]
      Hint: Do a proof by contradiction.

6. Prove theorem 5.1.20
   Hint: Just use the definitions of the various concepts involved.
5.2 Limit Theorems

5.2.1 Operations with Limits

The same theorem we proved for sequences also hold for functions. We list the theorem, and leave its proof as an exercise.

**Theorem 5.2.1** Assuming that \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist, the following results are true:

1. \( \lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \)

2. \( \lim_{x \to a} (f(x) g(x)) = \left( \lim_{x \to a} f(x) \right) \left( \lim_{x \to a} g(x) \right) \)

3. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \text{ as long as } \lim_{x \to a} g(x) \neq 0 \)

4. \( \lim_{x \to a} |f(x)| = \left| \lim_{x \to a} f(x) \right| \)

5. If \( f(x) \geq 0 \), then \( \lim_{x \to a} f(x) \geq 0 \)

6. If \( f(x) \geq g(x) \) then \( \lim_{x \to a} f(x) \geq \lim_{x \to a} g(x) \)

7. If \( f(x) \geq 0 \), then \( \lim_{x \to a} \sqrt{f(x)} = \sqrt{\lim_{x \to a} f(x)} \)

**Remark 5.2.2** The above results also hold when the limits are taken as \( x \to \pm \infty \).

**Remark 5.2.3** All the techniques learned in Calculus can be used here. These techniques include factoring, multiplying by the conjugate.

We look at a few examples to refresh the reader’s memory of some standard techniques.

**Example 5.2.4** Find \( \lim_{x \to 4} \frac{x^2 - 6x + 8}{x - 4} \)

Here, we note that both the numerator and denominator are approaching 0 as \( x \to 4 \). Since both the numerator and denominator are polynomials, we know we can factor \( x - 4 \).

\[
\lim_{x \to 4} \frac{x^2 - 6x + 8}{x - 4} = \lim_{x \to 4} \frac{(x - 4)(x - 2)}{x - 4}
\]

\[
= \lim_{x \to 4} (x - 2)
\]

\[
= 2
\]
5.2. LIMIT THEOREMS

Example 5.2.5 Find \( \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \)

Here, we note that both the numerator and denominator are approaching 0 as \( x \to 1 \). The fraction also contains a radical. A standard technique is to try to multiply by the conjugate.

\[
\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \to 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}
\]

Example 5.2.6 Find \( \lim_{x \to \infty} \frac{x^2 + 5x + 1}{2x^2 - 10} \)

When taking the limit as \( x \to \pm \infty \) of a rational function, a standard technique is to factor the term of highest degree from both the numerator and denominator.

\[
\lim_{x \to \infty} \frac{x^2 + 5x + 1}{2x^2 - 10} = \lim_{x \to \infty} \frac{x^2 \left(1 + \frac{5}{x} + \frac{1}{x^2}\right)}{x^2 \left(2 - \frac{10}{x^2}\right)} = \lim_{x \to \infty} \frac{1 + \frac{5}{x} + \frac{1}{x^2}}{2 - \frac{10}{x^2}} = 1\frac{1}{2}
\]

Remark 5.2.7 The above example is a special case of a more general result we give below.

Theorem 5.2.8 The limit as \( x \to \pm \infty \) of a rational function is the limit of the quotient of the terms of highest degree.

Proof. See problems. ■

Example 5.2.9 Find \( \lim_{x \to \infty} \frac{3x^2 + 1}{4x^3 + 2x + 1} \)

From the above theorem, we see that \( \lim_{x \to \infty} \frac{3x^2 + 1}{4x^3 + 2x + 1} = \lim_{x \to \infty} \frac{3x^2}{4x^3} = \lim_{x \to \infty} \frac{3}{4x} = 0 \).
5.2.2 Elementary Theorems

Theorems similar to those studied for sequences hold. We will leave the proof of most of these as an exercise.

**Theorem 5.2.10** If the limit of a function exists, then it is unique.

**Proof.** See exercises at the end of this section.

The next theorem relates the notion of limit of a function with the notion of limit of a sequence.

**Theorem 5.2.11** Suppose that \( \lim_{x \to a} f(x) = L \) and that \( \{x_n\} \) is a sequence of points such that \( \lim_{n \to \infty} x_n = a, x_n \neq a \forall n \). Put \( y_n = f(x_n) \). Then, \( \lim_{n \to \infty} y_n = L \)

**Proof.** Let \( \epsilon > 0 \) be given. Since \( \lim_{x \to a} f(x) = L \), we can find \( \delta \) such that \( 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon \). Since \( \lim_{n \to \infty} x_n = a \), we can find \( N \) such that \( n \geq N \implies |x_n - a| < \delta \). But in this case, it follows that \( |f(x_n) - L| < \epsilon \).

The converse of this theorem is also true.

**Theorem 5.2.12** If \( f \) is defined in a deleted neighborhood of \( a \) such that \( f(x_n) \to L \) for every sequence \( \{x_n\} \) such that \( x_n \to a \), then \( \lim_{x \to a} f(x) = L \).

**Proof.** See problems at the end of this section.

Like for sequences, if a function has a limit at a point, then it is bounded. However, we need to be a little bit more careful here. If \( \lim_{x \to a} f(x) = L \), then we are only given information about the behavior of \( f \) close to \( a \). Therefore, we can only draw conclusions about what happens to \( f \) as long as \( x \) is close to \( a \).

**Theorem 5.2.13** If \( \lim_{x \to a} f(x) = L \), then there exists a deleted neighborhood of \( a \) in which \( f \) is bounded.

**Proof.** We can find \( \delta \) such that \( 0 < |x - a| < \delta \implies |f(x) - L| < 1 \). By the triangle inequality, we have for such \( x \)’s

\[
|f(x)| - |L| \leq ||f(x)| - |L|| \\
\leq |f(x) - L| \\
< 1
\]

It follows that

\[
|f(x)| < 1 + |L|
\]

Therefore, \( f \) is bounded by \( 1 + |L| \) in \((a - \delta, a + \delta)\).

**Theorem 5.2.14** Suppose that \( f(x) \leq g(x) \leq h(x) \) in a deleted neighborhood of \( a \) and \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \) then \( \lim_{x \to a} g(x) = L \).

**Proof.** We do a direct proof here. In section 5.2.3, we’ll see another proof. Suppose the above inequality holds in \((a - \delta_1, a + \delta_1)\) for some \( \delta_1 > 0 \). Let \( \epsilon > 0 \)
be given. Choose $\delta_2$ such that $0 < |x - a| < \delta_2 \implies \epsilon - L < f(x) < \epsilon + L$. Choose $\delta_3$ such that $0 < |x - a| < \delta_3 \implies \epsilon - L < h(x) < \epsilon + L$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then, if $0 < |x - a| < \delta$, we have
\[
\epsilon - L < f(x) \leq g(x) \leq h(x) < \epsilon + L
\]
\[
\iff \epsilon - L < g(x) < \epsilon + L
\]
\[
\iff |g(x) - L| < \epsilon
\]
Therefore, $\lim_{x \to a} g(x) = L$. □

**Example 5.2.15** Show that $\lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right) = 0$.

### 5.2.3 Relationship Between the Limit of a Function and the Limit of a Sequence

Theorems 5.2.11 and 5.2.12 can be summarized in the theorem below.

**Theorem 5.2.16** Let $f$ be a function of one real variable defined in a deleted neighborhood of a real number $a$. The following conditions are equivalent.

1. $\lim_{x \to a} f(x) = L$

2. For every sequence $\{x_n\}$ such that $x_n \neq a$ and $x_n \to a$ we have $\lim_{n \to \infty} f(x_n) = L$.

**Proof.** We prove both directions.

- $(1 \implies 2)$. We assume that $\lim_{x \to a} f(x) = L$. Let $\{x_n\}$ be a sequence such that $x_n \neq a$ and $x_n \to a$. Then, the conclusion follows from theorem 5.2.11.

- $(2 \implies 1)$. This is theorem 5.2.12. □

This theorem provides the link between the limit of a function and the limit of a sequence. Using this theorem, we can prove the theorems about the limit of a function by using their counterpart for sequences. We illustrate this with another version of the proof of the squeeze theorem.

**Theorem 5.2.17** Suppose that $f(x) \leq g(x) \leq h(x)$ in a deleted neighborhood of $a$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x \to a} g(x) = L$.

**Proof.** To show that $\lim_{x \to a} g(x) = L$, we need to show that if $x_n$ is any sequence which converges to $a$, then $g(x_n)$ is a sequence which converges to $L$. Let $x_n$ be a sequence such that $x_n \to a$, $x_n \neq a$. Then, $f(x_n) \to L$ and $h(x_n) \to L$. Since $f(x_n) \leq g(x_n) \leq h(x_n)$ we can apply the squeeze theorem for sequences to conclude that $g(x_n) \to L$. □
Theorem 5.2.16 also gives us a convenient way to show that a limit of a function does not exist. We summarize how in the following corollary.

**Corollary 5.2.18** Let \( f \) be a function of one real variable defined in a deleted neighborhood of a real number \( a \). Then, \( \lim_{x \to a} f(x) \) does not exist if either of the following conditions holds:

1. There exist sequences \((x_n)\) and \((y_n)\) with \( x_n \neq a \) and \( y_n \neq a \) such that \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = a \) but \( \lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n) \).

2. There exists a sequence \((x_n)\) with \( x_n \neq a \) such that \( \lim_{n \to \infty} x_n = a \) but the sequence \( f(x_n) \) diverges.

We illustrate this corollary with an example.

**Example 5.2.19** Consider \( f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) given by \( f(x) = \sin \left( \frac{1}{x} \right) \). Show \( \lim_{x \to 0} f(x) \) does not exist.

Consider the sequences \( x_n = \frac{1}{2\pi n} \) and \( y_n = \frac{1}{2\pi n + \frac{\pi}{2}} \). Then, clearly, \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0 \). Yet, \( f(x_n) = \sin (2\pi n) = 0 \) thus \( \lim_{n \to \infty} f(x_n) = 0 \) but \( f(y_n) = \sin \left( 2\pi n + \frac{\pi}{2} \right) = 1 \) thus \( \lim_{n \to \infty} f(y_n) = 1 \).

**5.2.4 Exercises**

1. Prove theorem 5.2.8.
2. Prove theorem 5.2.10. Do both a direct proof and a proof using sequences.
3. Prove theorem 5.2.1. Do both a direct proof and a proof using sequences.
4. Evaluate the following limits:
   
   \[
   \begin{align*}
   (a) \quad & \lim_{x \to -2} \frac{x^2 - 4}{x + 2} \\
   (b) \quad & \lim_{x \to 3} \frac{x^3 - 27}{x - 3} \\
   (c) \quad & \lim_{x \to 1} \frac{x^n - 1}{x - 1} \quad \text{where } n \text{ is a positive integer.} \\
   (d) \quad & \lim_{x \to 1} \frac{x^n - 1}{x^n - 1} \\
   (e) \quad & \lim_{x \to 6} \frac{\sqrt{x - 2} - 2}{x - 6} 
   \end{align*}
   \]

5. Assuming you know that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \), compute the limits below:
5.2. LIMIT THEOREMS

(a) \( \lim_{x \to 0} \frac{\sin 2x}{x} \)

(b) \( \lim_{x \to 0} \frac{\sin 3x}{\sin 5x} \)

(c) \( \lim_{x \to 0} \frac{x}{\tan x} \)

(d) \( \lim_{x \to 0} \frac{\sin x}{\sqrt{x}} \)

6. Evaluate \( \lim_{x \to 0} \frac{\sqrt{(a + bx)(c + dx)} - \sqrt{ac}}{x} \)

7. Prove Theorem 5.2.12

8. Prove that if \( \lim_{x \to a} f(x) > 0 \) then there exists \( \delta > 0 \) such that \( f(x) > 0 \) for every \( x \in (a - \delta, a + \delta) \).

9. We say that a function \( f : I \to \mathbb{R} \) where \( I \subseteq \mathbb{R} \) is Lipschitz providing there exists a constant \( K > 0 \) such that

\[ |f(x) - f(y)| \leq K |x - y| \]

for \( x, y \in I \).

(a) Give an example of a Lipschitz function. You must show that the function in your example satisfies the required condition.

(b) Prove rigorously that if \( f \) is Lipschitz, then \( \lim_{x \to a} f(x) = f(a) \).
5.2.5 Hints for the Exercises

1. Prove theorem 5.2.8
   Hint: Write the expression of a typical rational function, then factor the term of highest degree from both the numerator and denominator.

2. Prove theorem 5.2.10
   Hint: The proof is similar to the same result for sequences.

3. Prove theorem 5.2.1
   Hint: Rewrite the result to prove using theorems 5.2.11 and 5.2.12 then use similar results for sequences.

4. Evaluate the following limits:
   (a) \( \lim_{x \to 2} \frac{x^2 - 4}{x + 2} \)
   Hint: Use simple algebra to factor some terms.
   (b) \( \lim_{x \to 3} \frac{x^3 - 27}{x - 3} \)
   Hint: Use simple algebra to factor some terms.
   (c) \( \lim_{x \to 1} \frac{x^n - 1}{x - 1} \) where \( n \) is a positive integer.
   Hint: Use simple algebra to factor some terms, remembering that \( x^n - 1 = (x - 1)(1 + x + x^2 + \ldots + x^{n-1}) \)
   (d) \( \lim_{x \to 1} \frac{x^n - 1}{x^m - 1} \)
   Hint: Use simple algebra to factor some terms, remembering that \( x^n - 1 = (x - 1)(1 + x + x^2 + \ldots + x^{n-1}) \)
   (e) \( \lim_{x \to 6} \frac{\sqrt{x - 2} - 2}{x - 6} \)
   Hint: Use the conjugate.

5. Assuming you know that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \), compute the limits below:
   (a) \( \lim_{x \to 0} \frac{\sin 2x}{x} \)
   Hint: Use the fact that \( x \to 0 \iff 2x \to 0 \).
   (b) \( \lim_{x \to 0} \frac{\sin 3x}{x} \)
   Hint: Remembering the previous problem, rewrite this problem so it looks like \( \frac{\sin x}{x} \) so you can use the given result.
   (c) \( \lim_{x \to 0} \frac{x}{\tan x} \)
   Hint: Use the definition of \( \tan x \).
(d) \[ \lim_{x \to 0} \frac{\sin x}{\sqrt{x}} \]
Hint: In order to use the given result, you need to rewrite this so that there is an \( x \) in the denominator.

6. Evaluate \( \lim_{x \to 0} \frac{\sqrt{(a + bx)(c + dx)} - \sqrt{ac}}{x} \)
Hint: Use the conjugate.

7. Prove Theorem 5.2.12
Hint: Using the definition of limits, try a proof by contradiction.

8. Prove that if \( \lim_{x \to a} f(x) > 0 \) then there exists \( \delta > 0 \) such that \( f(x) > 0 \) for every \( x \in (a - \delta, a + \delta) \).
Hint: Using the definition of limits, pick an appropriate \( \varepsilon \) to have the desired result.

9. We say that a function \( f : I \to \mathbb{R} \) where \( I \subseteq \mathbb{R} \) is Lipschitz providing there exists a constant \( K > 0 \) such that
\[ |f(x) - f(y)| \leq K |x - y| \]
for \( x, y \in I \).

(a) Give an example of a Lipschitz function. You must show that the function in your example satisfies the required condition.
Hint: Try to understand the meaning of this condition. To help you do that, think of the geometric meaning of the quantity \( \frac{f(x) - f(y)}{x - y} \).

(b) Prove rigorously that if \( f \) is Lipschitz, then \( \lim_{x \to a} f(x) = f(a) \).
Hint: Simply use the definition of limit.
Bibliography


