2.4 Sequences, Sequences of Sets

2.4.1 Sequences

Definition 2.4.1 (sequence) Let $S \subseteq \mathbb{R}$.

1. A sequence in $S$ is a function $f : K \rightarrow S$ where $K = \{n \in \mathbb{N} : n \geq n_0 \text{ for some } n_0 \in \mathbb{N}\}$.

2. For each $n \in K$, we let $x_n = f(n)$. $x_n$ is called the $n^{th}$ term of the sequence $f$.

For convenience, we usually denote a sequence by $\{x_n\}_{n=n_0}^{\infty}$ rather than $f$. Some texts also use $(x_n)_{n=n_0}^{\infty}$. The starting point, $n_0$ is usually 1 in which case, we simply write $\{x_n\}$ to denote a sequence. In theory, the starting point does not have to be 1. However, it is understood that whatever the starting point is, the elements $x_n$ should be defined for any $n \geq n_0$. For example, if the general term of a sequence is $x_n = \frac{2n}{n-4}$, then, we must have $n_0 \geq 5$. We can think of a sequence as a list of numbers. In this case, a sequence will look like: $\{x_n\} = \{x_1, x_2, x_3, \ldots\}$.

A sequence can be given different ways.

1. List the elements. For example, $\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \right\}$. From the elements listed, the pattern should be clear.

2. Give a formula to generate the terms. For example, $x_n = (-1)^n \frac{2^n}{n!}$. If the starting point is not specified, we use the smallest value of $n$ which will work.

3. A sequence can be given recursively. The starting value of the sequence is given. Then, a formula to generate the $n^{th}$ term from one or more previous terms. For example, we could define a sequence by giving:

$$
x_1 = 2 \\
x_{n+1} = \frac{1}{2} (x_n + 6)
$$

Another example is the **Fibonacci sequence** defined by:

$$
x_1 = 1, \ x_2 = 1 \\
x_n = x_{n-1} + x_{n-2} \text{ for } n \geq 3
$$

Like a function, a sequence can be plotted. However, since the domain is a subset of $\mathbb{Z}$, the plot will consist of dots instead of a continuous curve.

Since a sequence is defined as a function, everything we defined for functions (bounds, supremum, infimum, ...) also applies to sequences. We restate those definitions for convenience.
CHAPTER 2. IMPORTANT PROPERTIES OF $\mathbb{R}$

Definition 2.4.2 (Bounded Sequence) A sequence $(x_n)$ is said to be bounded above if its range is bounded above. It is bounded below if its range is bounded below. It is bounded if its range is bounded. If the domain of $(x_n)$ is $\{n \in \mathbb{Z} : n \geq k \text{ for some integer } k\}$ then the above definition simply state that the set $\{x_n : n \geq k\}$ must be bounded above, below or both.

Definition 2.4.3 (One-to-one Sequences) A sequence $(x_n)$ is said to be one-to-one if whenever $n \neq m$ then $x_n \neq x_m$.

Definition 2.4.4 (Monotone Sequences) Let $(x_n)$ be a sequence.

1. $(x_n)$ is said to be increasing if $x_n \leq x_{n+1}$ for every $n$ in the domain of the sequence. If we have $x_n < x_{n+1}$, we say the sequence is strictly increasing.

2. $(x_n)$ is said to be decreasing if $x_n \geq x_{n+1}$ for every $n$ in the domain of the sequence.

3. A sequence that is either increasing or decreasing is said to be monotone. If it is either strictly increasing or strictly decreasing, we say it is strictly monotone.

2.4.2 Sequences of Sets and Indexed Families of Sets

Sequences of sets are similar to sequences of real numbers with the exception that the terms of the sequence are sets. The notation used is also similar. We also generalize the definition to indexed families of sets where the index set is not a subset of $\mathbb{Z}$ but any set. More precisely, we have:

Definition 2.4.5 Let $X \subseteq \mathbb{R}$, $X \neq \emptyset$ and let $K = \{n \in \mathbb{N} : n \geq n_0 \text{ for some } n_0 \in \mathbb{N}\}$. A sequence of subsets of $X$ is a function $f : K \rightarrow P(X)$, the power set of $X$. For each $n \in K$, we let $A_n = f(n)$ be a subset of $X$ that is an element of $P(X)$.

Example 2.4.6 Consider $\{\mathbb{N}_n\}_{n=1}^{\infty}$ where $\mathbb{N}_n = \{1, 2, 3, ..., n\}$. $\{\mathbb{N}_n\}_{n=1}^{\infty}$ is a sequence of subsets of $\mathbb{N}$.

Example 2.4.7 For each $n \in \mathbb{N}$, define $I_n = \left\{x \in \mathbb{R} : 0 < x < \frac{1}{n}\right\} = \left(0, \frac{1}{n}\right)$. $\{I_n\}_{n=1}^{\infty}$ is a sequence of subsets of $\mathbb{R}$.

The index we use for the subsets does not have to be an integer. It can be an element of any set. In this case, instead of calling it a sequence of sets, we call it an indexed family of sets. We give a precise definition.

Definition 2.4.8 Let $A$ and $X$ be non-empty sets. An indexed family of subsets of $X$ with index set $A$ is a function $f : A \rightarrow P(X)$ (the power set of $X$). Like for sequences, if $f : A \rightarrow P(X)$, then for each $\alpha \in A$, we let $E_\alpha = f(\alpha)$. We use a notation similar to sequences that is we denote the indexed family $\{E_\alpha\}_{\alpha \in A}$.
Remark 2.4.9 A sequence of sets is just an indexed family of sets with \( \mathbb{N} \) as index set.

Example 2.4.10 For each \( x \in (0, 1) \), define \( E_x = \{ r \in \mathbb{Q} : 0 \leq r < x \} \). Then, \( \{E_x\}_{x \in (0, 1)} \) is an indexed family of subsets of \( \mathbb{Q} \). The index set is \( (0, 1) \).

The remainder of this section deals with sequences of sets, though the results and definitions given can be extended to indexed families of subsets.

Definition 2.4.11 (Union and Intersection of a Sequence of Subsets) Let \( \{A_n\} \) be a sequence of subsets of a set \( X \).

1. We define
   \[
   \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = \{ x \in X : x \in A_i \text{ for some natural number } 1 \leq i \leq n \}
   
   Similarly, we define the union of the entire sequence by
   \[
   \bigcup_{i=1}^{\infty} A_i = \{ x \in X : x \in A_i \text{ for some natural number } i \}
   
   2. Similarly, we define
   \[
   \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n = \{ x \in X : x \in A_i \text{ for every natural number } 1 \leq i \leq n \}
   
   and
   \[
   \bigcap_{i=1}^{\infty} A_i = \{ x \in X : x \in A_i \text{ for every natural number } i \}
   
   Example 2.4.12 Consider \( \{N_n\}_{n=1}^{\infty} \) where \( N_n = \{1, 2, 3, \ldots, n\} \). Then, \( \bigcup_{i=1}^{\infty} N_i = \mathbb{N} \) and \( \bigcap_{i=1}^{\infty} N_i = \{1\} \).

Example 2.4.13 For each \( n \in \mathbb{N} \), define \( I_n = \left\{ x \in \mathbb{R} : 0 < x < \frac{1}{n} \right\} \). First, we prove that
   \[
   \bigcap_{i=1}^{\infty} I_i = \emptyset
   
   \]
If this were not the case, that is if we had \( x \in \bigcap_{i=1}^{\infty} I_i \) this would mean that no matter what \( n \) is, \( x < \frac{1}{n} \) or \( \frac{1}{n} > x \) which contradicts the Archimedean principle (theorem 2.2.5). We next show that
\[
\bigcup_{i=1}^{\infty} I_i = I_1
\]
This because \( I_n \subseteq I_1 \) for any \( n \geq 1 \).

Results about finite intersection and union of sets remain true in this setting. In other words, we have the equivalent of theorems [1.1.16] (distributive properties), [1.1.20] (De Morgan’s laws), [1.2.33] (direct image of a set) and [1.2.35] (inverse image of a set). We list the theorem here but leave their proof as exercises.

In the next few theorems, think of \( X \) as \( \mathbb{R} \) since this is real analysis.

**Theorem 2.4.14 (Distributive Laws)** Let \( E_n \) and \( E \) be subsets of a set \( X \). Then,
1. \( E \cap \left( \bigcup_{i=1}^{\infty} E_i \right) = \bigcup_{i=1}^{\infty} (E \cap E_i) \)
2. \( E \cup \left( \bigcap_{i=1}^{\infty} E_i \right) = \bigcap_{i=1}^{\infty} (E \cup E_i) \)

**Proof.** See problems.

**Theorem 2.4.15 (De Morgan’s Laws)** Let \( \{E_n\} \) be a sequence of subsets of \( X \). Then,
1. \( \left( \bigcup_{i=1}^{\infty} E_i \right)^c = \bigcap_{i=1}^{\infty} E_i^c \)
2. \( \left( \bigcap_{i=1}^{\infty} E_i \right)^c = \bigcup_{i=1}^{\infty} E_i^c \)

**Proof.** See problems.

**Theorem 2.4.16** Let \( f : X \to Y \)

1. If \( \{E_n\} \) is a sequence of subsets of \( X \), then
\[
f \left( \bigcup_{i=1}^{\infty} E_i \right) = \bigcup_{i=1}^{\infty} f(E_i)
\]
\[
f \left( \bigcap_{i=1}^{\infty} E_i \right) \subseteq \bigcap_{i=1}^{\infty} f(E_i)
\]
2. If \( \{G_n\} \) is a sequence of subsets of \( Y \), then
\[
 f^{-1}\left( \bigcup_{i=1}^{\infty} G_i \right) = \bigcup_{i=1}^{\infty} f^{-1}(G_i)
\]
\[
 f^{-1}\left( \bigcap_{i=1}^{\infty} G_i \right) = \bigcap_{i=1}^{\infty} f^{-1}(G_i)
\]

**Proof.** See problems. ■

**Definition 2.4.17** (Contracting and Expanding Sequences of Sets) Suppose that \( \{A_n\} \) is a sequence of sets.

1. We say that \( \{A_n\} \) is **expanding** if we have \( A_n \subseteq A_{n+1} \) for every \( n \) in the domain of the sequence.

2. We say that \( \{A_n\} \) is **contracting** if we have \( A_{n+1} \subseteq A_n \) for every \( n \) in the domain of the sequence.

**Example 2.4.18** Consider the sequence of sets \( \{J_n\} \) where \( J_n = \left( -\frac{1}{n}, \frac{1}{n} \right) \).

\( J_n \) is a contracting sequence as \( J_{n+1} = \left( -\frac{1}{n+1}, \frac{1}{n+1} \right) \subseteq J_n \).

**Example 2.4.19** Consider the sequence of sets \( \{K_n\} \) where \( K_n = (-n, n) \). \( K_n \) is an expanding sequence as \( K_n \subseteq K_{n+1} = (-n+1, n+1) \).

**2.4.3 Exercises**

1. Let \( I_n = \left[ \frac{1}{n}, 1 \right] \). Evaluate
\[
 \bigcup_{n=1}^{\infty} I_n
\]

2. Let \( I_n = \left[ 1 + \frac{1}{n}, 5 - \frac{2}{n} \right] \). Evaluate
\[
 \bigcup_{n=1}^{\infty} I_n
\]

3. Let \( I_n = \left( 1 - \frac{1}{n}, 2 + \frac{1}{n} \right) \). Evaluate
\[
 \bigcap_{n=1}^{\infty} I_n
\]
4. Prove theorem 2.4.14

5. Prove theorem 2.4.15

6. Prove theorem 2.4.16. In particular, for part 2 of the theorem, explain why we do not have equality. Give a necessary condition to have equality and justify your answer.

7. Let \( \{A_n\} \) be an expanding sequence of subsets of \( \mathbb{R} \). Prove that \( \{\mathbb{R} \setminus A_n\} \) is a contracting sequence.

8. Let \( \{A_n\} \) be a sequence of sets.

   (a) Prove that if we define
   \[
   B_n = \bigcup_{i=n}^{\infty} A_i
   \]
   for each \( n \), the sequence \( \{B_n\} \) is a contracting sequence of sets.

   (b) Prove that if we define
   \[
   B_n = \bigcap_{i=n}^{\infty} A_i
   \]
   for each \( n \), the sequence \( \{B_n\} \) is an expanding sequence of sets.

   (c) Prove that if we define
   \[
   B_n = \bigcap_{i=1}^{n} A_i
   \]
   for each \( n \), the sequence \( \{B_n\} \) is an expanding sequence of sets.

   (d) Prove that if we define
   \[
   B_n = \bigcup_{i=1}^{n} A_i
   \]
   for each \( n \), the sequence \( \{B_n\} \) is an expanding sequence of sets.
2.4. Hints for the Exercises

For all the problems, remember to first clearly state what you have to prove or do.

1. Let \( I_n = \left( \frac{1}{n}, 1 \right) \). Evaluate
\[
\bigcup_{n=1}^{\infty} I_n
\]

Hint: Evaluate \( I_n \) for a few values of \( n \) to see what is happening. You should then be able to conjecture what the answer is. You then have to prove the conjecture which will involve proving two sets are equal. You will have to use the Archimedean principle.

2. Let \( I_n = \left( 1 + \frac{1}{n}, 5 - \frac{2}{n} \right) \). Evaluate
\[
\bigcup_{n=1}^{\infty} I_n
\]

Hint: Evaluate \( I_n \) for a few values of \( n \) to see what is happening. You should then be able to conjecture what the answer is. You then have to prove the conjecture which will involve proving two sets are equal. You will have to use the Archimedean principle.

3. Let \( I_n = \left( 1 - \frac{1}{n}, 2 + \frac{1}{n} \right) \). Evaluate
\[
\bigcap_{n=1}^{\infty} I_n
\]

Hint: Evaluate \( I_n \) for a few values of \( n \) to see what is happening. You should then be able to conjecture what the answer is. You then have to prove the conjecture which will involve proving two sets are equal. You will have to use the Archimedean principle.

4. Prove theorem 2.4.14
   Hint: nothing hard, just use definitions and standard techniques to show two sets are equal.

5. Prove theorem 2.4.15
   Hint: nothing hard, just use definitions and standard techniques to show two sets are equal.

6. Prove theorem 2.4.16 In particular, for part 2 of the theorem, explain why we do not have equality. Give a necessary condition to have equality.
and justify your answer.
Hint: nothing hard, just use definitions and standard techniques to show two sets are equal.

7. Let \( \{A_n\} \) be an expanding sequence of subsets of \( \mathbb{R} \). Prove that \( \{\mathbb{R} \setminus A_n\} \) is a contracting sequence.
Hint: no difficulty, just use definitions. Make sure to first clearly write what you have to prove.

8. Let \( \{A_n\} \) be a sequence of sets.

(a) Prove that if we define
\[
B_n = \bigcup_{i=n}^{\infty} A_i
\]
for each \( n \), the sequence \( \{B_n\} \) is a contracting sequence of sets.
Hint: Make sure you understand how this set \( B_n \) is defined by making your own examples. You prove the result by just using definitions. Again, first state clearly what you have to prove.

(b) Prove that if we define
\[
B_n = \bigcap_{i=n}^{\infty} A_i
\]
for each \( n \), the sequence \( \{B_n\} \) is an expanding sequence of sets.
Hint: Make sure you understand how this set \( B_n \) is defined by making your own examples. You prove the result by just using definitions. Again, first state clearly what you have to prove.

(c) Prove that if we define
\[
B_n = \bigcap_{i=1}^{n} A_i
\]
for each \( n \), the sequence \( \{B_n\} \) is a contracting sequence of sets.
Hint: Make sure you understand how this set \( B_n \) is defined by making your own examples. You prove the result by just using definitions. Again, first state clearly what you have to prove.

(d) Prove that if we define
\[
B_n = \bigcup_{i=1}^{n} A_i
\]
for each \( n \), the sequence \( \{B_n\} \) is an expanding sequence of sets.
Hint: Make sure you understand how this set \( B_n \) is defined by making your own examples. You prove the result by just using definitions. Again, first state clearly what you have to prove.
Bibliography


