Sequences of Sets, Indexed Families of Sets

Philippe B. Laval

KSU

Today
We extend the definition of sequences studied in Calculus to sequences of sets and indexed families of sets. We then prove that the main properties of the operations on two sets extend to sequences of sets.
Sequences: Quick review

**Definition (sequence)**

Let $S \subseteq \mathbb{R}$.

1. A **sequence** in $S$ is a function $f : K \to S$ where $K = \{n \in \mathbb{N} : n \geq n_0 \text{ for some } n_0 \in \mathbb{N}\}$.

2. For each $n \in K$, we let $x_n = f(n)$. $x_n$ is called the $n^{th}$ term of the sequence $f$.

For convenience, we usually denote a sequence by $\{x_n\}_{n=n_0}^{\infty}$ rather than $f$. Some texts also use $(x_n)_{n=n_0}^{\infty}$. The starting point, $n_0$ is usually 1 in which case, we simply write $\{x_n\}$ or $(x_n)$ to denote a sequence. In theory, the starting point does not have to be 1. However, it is understood that whatever the starting point is, the elements $x_n$ should be defined for any $n \geq n_0$. 
A sequence can be given different ways.

1. List the elements.
2. Give a formula to generate the terms.
3. A sequence can be given recursively.

Like a function, a sequence can be plotted. However, since the domain is a subset of \( \mathbb{Z} \), the plot will consist of dots instead of a continuous curve. We now focus on sequences of sets and indexed families of sets.
Sequences of Sets

Sequences of sets are similar to sequences of real numbers but the terms of the sequence are sets. The notation used is also similar.

**Definition**

Let $X \subseteq \mathbb{R}$, $X \neq \emptyset$ and let $K = \{ n \in \mathbb{N} : n \geq n_0 \text{ for some } n_0 \in \mathbb{N} \}$. A sequence of subsets of $X$ is a function $f : K \rightarrow P(X)$, the power set of $X$. For each $n \in K$, we let $A_n = f(n)$ be a subset of $X$ that is an element of $P(x)$.

**Example**

Consider $\{ \mathbb{N}_n \}_{n=1}^{\infty}$ where $\mathbb{N}_n = \{1, 2, 3, ..., n\}$. $\{ \mathbb{N}_n \}_{n=1}^{\infty}$ is a sequence of subsets of $\mathbb{N}$.

**Example**

For each $n \in \mathbb{N}$, define $I_n = \left\{ x \in \mathbb{R} : 0 < x < \frac{1}{n} \right\} = \left( 0, \frac{1}{n} \right)$. $\{ I_n \}_{n=1}^{\infty}$ is a sequence of subsets of $\mathbb{R}$.
Indexed Families of Sets

The index we use for the subsets does not have to be an integer. It can be an element of any set. In this case, instead of calling it a sequence of sets, we call it an indexed family of sets. We give a precise definition.

**Definition**

Let $A$ and $X$ be non-empty sets. An **indexed family** of subsets of $X$ with index set $A$ is a function $f : A \rightarrow P(X)$. Like for sequences, if $f : A \rightarrow P(X)$, then for each $\alpha \in A$, we let $E_\alpha = f(\alpha)$. We use a notation similar to sequences that is we denote the indexed family $\{E_\alpha\}_{\alpha \in A}$.

**Remark**

A sequence of sets is just an indexed family of sets with $\mathbb{N}$ as index set.

**Example**

For each $x \in (0, 1)$, define $E_x = \{r \in \mathbb{Q} : 0 \leq r < x\}$. Then, $\{E_x\}_{x \in (0, 1)}$ is an indexed family of subsets of $\mathbb{Q}$. The index set is $(0, 1)$. 
Union and Intersection of a Sequence of Subsets

The remainder of this section deals with sequences of sets, though the results and definitions given can be extended to indexed families of subsets.

**Definition (Union of a Sequence of Subsets)**

Let \( \{A_n\} \) be a sequence of subsets of a set \( X \). We define

\[
\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n
\]

\[
= \{ x \in X : x \in A_i \text{ for some natural number } 1 \leq i \leq n \}
\]

Similarly, we define the union of the entire sequence by

\[
\bigcup_{i=1}^{\infty} A_i = \{ x \in X : x \in A_i \text{ for some natural number } i \}
\]
Definition (Intersection of a Sequence of Subsets)

Let \( \{ A_n \} \) be a sequence of subsets of a set \( X \). We define

\[
\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap ... \cap A_n = \{ x \in X : x \in A_i \text{ for every natural number } 1 \leq i \leq n \}
\]

and

\[
\bigcap_{i=1}^{\infty} A_i = \{ x \in X : x \in A_i \text{ for every natural number } i \}
\]
Example

Consider \( \{N_n\}_{n=1}^{\infty} \) where \( N_n = \{1, 2, 3, \ldots, n\} \). Find \( \bigcup_{i=1}^{\infty} N_i \) and \( \bigcap_{i=1}^{\infty} N_i \).

Example

For each \( n \in \mathbb{N} \), define \( l_n = \left\{ x \in \mathbb{R} : 0 < x < \frac{1}{n} \right\} \). Prove that \( \bigcap_{i=1}^{\infty} l_i = \emptyset \)

and \( \bigcup_{i=1}^{\infty} l_i = l_1 \).
Theorem (Distributive Laws)

Let $E_n$ and $E$ be subsets of a set $X$. Then,

1. $E \cap \left( \bigcup_{i=1}^{\infty} E_i \right) = \bigcup_{i=1}^{\infty} (E \cap E_i)$

2. $E \cup \left( \bigcap_{i=1}^{\infty} E_i \right) = \bigcap_{i=1}^{\infty} (E \cup E_i)$
Theorem (De Morgan’s Laws)

Let \( \{E_n\} \) be a sequence of subsets of \( X \). Then,

1. \( \left( \bigcup_{i=1}^{\infty} E_i \right)^c = \bigcap_{i=1}^{\infty} E_i^c \)
2. \( \left( \bigcap_{i=1}^{\infty} E_i \right)^c = \bigcup_{i=1}^{\infty} E_i^c \)
Theorems

**Theorem**

Let \( f : X \rightarrow Y \)

1. If \( \{E_n\} \) is a sequence of subsets of \( X \), then
   \[
   f \left( \bigcup_{i=1}^{\infty} E_i \right) = \bigcup_{i=1}^{\infty} f(E_i) \quad \text{and} \quad f \left( \bigcap_{i=1}^{\infty} E_i \right) \subseteq \bigcap_{i=1}^{\infty} f(E_i)
   \]

2. If \( \{G_n\} \) is a sequence of subsets of \( Y \), then
   \[
   f^{-1} \left( \bigcup_{i=1}^{\infty} G_i \right) = \bigcup_{i=1}^{\infty} f^{-1}(G_i) \quad \text{and} \quad f^{-1} \left( \bigcap_{i=1}^{\infty} G_i \right) = \bigcap_{i=1}^{\infty} f^{-1}(G_i)
   \]
Contracting and Expanding Sequences

Definition (Contracting and Expanding Sequences of Sets)

Suppose that \((A_n)\) is a sequence of sets.

1. We say that \((A_n)\) is **expanding** if we have \(A_n \subseteq A_{n+1}\) for every \(n\) in the domain of the sequence.

2. We say that \((A_n)\) is **contracting** if we have \(A_{n+1} \subseteq A_n\) for every \(n\) in the domain of the sequence.

Example

Is the sequence of sets \((J_n)\) where \(J_n = \left( \frac{-1}{n}, \frac{1}{n} \right)\). \(J_n\) contracting or expanding?

Example

Is the sequence of sets \((K_n)\) where \(K_n = (-n, n)\). \(K_n\) contracting or expanding?
Exercises

See the problems at the end of my notes on sequences of sets.