2.9 The Extended Real Number System

2.9.1 Introduction

In mathematics, when we want to express the idea that a quantity can be made as large as one wishes, we simply say that this quantity is infinite ($\infty$). For example, you may remember from Calculus

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

In calculus, you were also told that $\infty$ was not a number and that you could not do arithmetic with infinity. Sometimes, it turns out that it is useful to treat $\infty$ and $-\infty$ as if they were numbers. They make the statement of some theorems and properties easier to write. They also decrease the number of cases to consider when proving results.

**Definition 227 (Extended Real Number System)** In treating $\infty$ and $-\infty$ as numbers, we are extending the real number system. What we have is

$$\mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$$

This is called the **extended real number system**. It is sometimes denoted $\mathbb{R}^\#$

In the extended real number system, we have the usual order $-\infty < \infty$, and for any real number $x$, $-\infty < x < \infty$.

2.9.2 Arithmetic in the Extended Real Number System

Many of the operations we do with real numbers can be extended to the extended real number system, but not all.

$$\infty + \infty = \infty \times \infty = (-\infty)(-\infty) = \infty$$

$$-\infty - \infty = (-\infty)\infty = \infty(-\infty) = -\infty$$

If $x$ is any real number, then

$$\infty + x = x + \infty = \infty$$

$$-\infty + x = x - \infty = -\infty$$

$$\frac{x}{\infty} = \frac{x}{-\infty} = 0$$

$$\infty \times x = x \times \infty = \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases}$$

$$(-\infty) \times x = x \times (-\infty) = \begin{cases} -\infty & \text{if } x > 0 \\ \infty & \text{if } x < 0 \end{cases}$$
However, the following are still indeterminate forms. Their behavior is unpredictable. Finding what they are equal to requires more advanced techniques such as l’Hôpital’s rule.

\[ -\infty + \infty \text{ and } \infty - \infty \]

\[ 0 \times \infty \text{ and } \infty \times 0 \]

\[ \infty \]

\[ \infty \]

### 2.9.3 Suprema and Infima

If a set is unbounded above in \( \mathbb{R} \), we can think of \( \infty \) as being an upper bound of this set in the extended real number system. The same applies for lower bounds and \( -\infty \).

### 2.9.4 Exercises

1. #1 on page 89 (end of section 5.14) in Lewin’s book.
2. #2 on page 89 (end of section 5.14) in Lewin’s book.
3. #3 on page 89 (end of section 5.14) in Lewin’s book.