

8.2 Testing Series with Positive Terms

This section presents some of the tests which can be used to test series with positive terms for convergence.

8.2.1 The Integral Test

This test compares a series to an integral (improper). It establishes that under certain conditions, convergence of a series can be established by studying the convergence of an improper integral. It even goes further. The value of the integral can also be used to approximate the infinite series. However, unlike for geometric series, this test will not give us an exact value for convergent infinite series.

Theorem 8.2.1 (Integral test) If \( f \) is a continuous, positive and decreasing function on \([1, \infty)\) and \( a_n = f(n) \) then \( \sum_{i=1}^{\infty} a_i \) converges if and only if \( \int_1^{\infty} f(x) \, dx \) converges.

Proof. From the given conditions, we see \( \sum_{i=2}^{n} a_i \) is a lower Riemann sum for \( f \) and \( \sum_{i=1}^{n-1} a_i \) is an upper Riemann sum on \([1, n] \), \( \forall n \in \mathbb{N} \). Hence, we have

\[
\sum_{i=2}^{n} a_i \leq \int_1^{n} f(x) \, dx \leq \sum_{i=1}^{n-1} a_i \tag{8.1}
\]

If \( \int_1^{\infty} f(x) \, dx \) converges, then by the first half of the inequality in 8.1 the partial sums of \( \sum_{i=2}^{\infty} a_i \) are bounded above so, by theorem 8.1.17, \( \sum_{i=2}^{\infty} a_i \) converges, hence \( \sum_{i=1}^{\infty} a_i \) converges. If \( \int_1^{\infty} f(x) \, dx \) diverges, then by the second half of the inequality in 8.1 the partial sums of \( \sum_{i=1}^{\infty} a_i \) are unbounded above so, by theorem 8.1.17, \( \sum_{i=1}^{\infty} a_i \) diverges.

Remark 8.2.2 The conditions in the integral test do not have to hold starting at \( n = 1 \), as long as they hold from some point on.

Remark 8.2.3 Before applying this test, make sure that the series satisfies all the conditions of the theorem.

Remark 8.2.4 The theorem does not indicate what the series converges to. You should not assume that the value of the integral and the series are the same.

Example 8.2.5 Test \( \sum_{n=1}^{\infty} \frac{\ln n}{n} \) for convergence.

The function defining the general term of this series is \( f(x) = \frac{\ln x}{x} \). It is
8.2. TESTING SERIES WITH POSITIVE TERMS

continuous and positive on \([1, \infty)\). To determine if it is decreasing, we study the sign of its derivative. \(f'(x) = \frac{1 - \ln x}{x^2}\). Since the denominator is a square, it is always positive. So the sign of the derivative is uniquely determined by its numerator.

\[
f'(x) < 0 \\
1 - \ln x < 0 \\
1 < \ln x \\
\ln x > 1 \\
x > e
\]

Therefore, \(f\) is not decreasing on \([1, \infty)\) but it is on \([3, \infty)\), which is good enough. Remember, this property only needs to hold from some point on, which it does here. Having verified that all the hypotheses of the theorem were satisfied, we can now use the integral test. We study the convergence of \(\int_3^\infty \frac{\ln x}{x} \, dx\). This is an improper integral.

\[
\int_3^\infty \frac{\ln x}{x} \, dx = \lim_{t \to \infty} \int_3^t \frac{\ln x}{x} \, dx \\
= \lim_{t \to \infty} \left[ \frac{(\ln t)^2}{2} - \frac{(\ln 3)^2}{2} \right] \\
= \infty
\]

The integral diverges, therefore the series diverges.

**Remark 8.2.6** The above integral was evaluated using the substitution \(u = \ln x\) therefore \(du = \frac{dx}{x}\).

**Example 8.2.7** Test \(\sum_{i=1}^{\infty} \frac{1}{i^p}\) for convergence.

If \(p < 0\), then \(\lim_{i \to \infty} \frac{1}{i^p} = \infty\), so the series diverges. If \(p = 0\), then \(\frac{1}{i^p} = \frac{1}{i^0} = 1\). Hence, \(\lim_{i \to \infty} \frac{1}{i^p} = 1\). The series also diverges. If \(p > 0\), then the function \(f(x) = \frac{1}{x^p}\) is continuous, decreasing (since its denominator is increasing) and positive on \([1, \infty)\). So, instead of studying the convergence of the series, we study the convergence of the integral \(\int_1^\infty \frac{dx}{x^p}\). From a theorem about improper integrals, we know that this integral converges if \(p > 1\) and diverges otherwise that is in this case when \(0 < p \leq 1\). Combining all the cases, we see that the series converges when \(p > 1\) and diverges otherwise.
The above example is very important. Its result should be remembered. Series which appear in the above example have a special name.

**Definition 8.2.8** A p-series is a series of the form \( \sum_{i=1}^{\infty} \frac{1}{i^p} \).

**Example 8.2.9** Here are some examples of p-series.

1. \( \sum_{i=1}^{\infty} \frac{1}{i^2} \) \((p = 2)\)
2. \( \sum_{i=1}^{\infty} \frac{1}{\sqrt{i}} \) \((p = \frac{1}{2})\)
3. \( \sum_{i=1}^{\infty} \frac{1}{i^{3/2}} \) \((p = \frac{3}{2})\)

In example 8.2.7 we proved the following:

**Theorem 8.2.10** A p-series converges if \( p > 1 \). It diverges otherwise.

**Example 8.2.11** Test \( \sum_{i=1}^{\infty} \frac{1}{i^{2}} \) for convergence.

This is a p-series with \( p = 2 \), therefore it converges.

**Example 8.2.12** Test \( \sum_{i=1}^{\infty} \frac{1}{\sqrt{i}} \) for convergence.

This is a p-series with \( p = \frac{1}{2} \), therefore it diverges.

### 8.2.2 Comparison Tests

The idea here is to compare an unknown series to a known series. Series such as the harmonic series, geometric series and p-series are used a lot. The question is, given an unknown series, how do we know what to compare it to. Series which look like a p-series or a geometric series should be compared with such series.

For example, \( \sum_{i=1}^{\infty} \frac{1}{n^2 + 1} \) should be compared with \( \sum_{i=1}^{\infty} \frac{1}{n^2} \) because for large \( n \), \( n^2 + 1 \approx n^2 \), thus \( \frac{1}{n^2 + 1} \approx \frac{1}{n^2} \). Similarly, \( \sum_{i=1}^{\infty} \frac{1}{2^n - 1} \) should be compared to \( \sum_{i=1}^{\infty} \frac{1}{2^n} \).

In addition, the following facts are often used and should be remembered.

- If \( a, b, c \) are positive numbers then \( b > c \) implies that \( \frac{a}{b} < \frac{a}{c} \)
- \( \ln x < x < e^x \)

One of two comparison tests can be used:
8.2. TESTING SERIES WITH POSITIVE TERMS

Theorem 8.2.13 (Standard comparison test) Suppose that \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are two series with positive terms such that \( a_n \leq b_n \) \( \forall n \in \mathbb{N} \). Then,

1. If \( \sum_{n=1}^{\infty} b_n \) converges, so does \( \sum_{n=1}^{\infty} a_n \) (i.e. if the series with larger terms converges, so will the series with smaller terms).

2. If \( \sum_{n=1}^{\infty} a_n \) diverges, so does \( \sum_{n=1}^{\infty} b_n \) (i.e. if the series with smaller terms diverges, so will the series with larger terms).

Proof. We prove each part separately.

1. If \( a_n \leq b_n \) \( \forall n \in \mathbb{N} \) then \( \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i \). So, if \( \sum_{n=1}^{\infty} b_n \) converges, the partial sums of \( \sum_{n=1}^{\infty} a_n \) are bounded above by \( \sum_{n=1}^{\infty} b_n \). Hence, by theorem 8.1.17, \( \sum_{n=1}^{\infty} a_n \) converges.

2. Clearly, this is the contrapositive of part 1.

Intuitively this theorem should make sense. Since we have series of positive terms, either the series is finite, in which case it converges or it is infinite, in which case it diverges. So, if \( 0 \leq a_n \leq b_n \) then

\[
0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n
\]

So, we see that if the series of larger terms, \( \sum_{n=1}^{\infty} b_n \) converges that is is finite, then \( \sum_{n=1}^{\infty} a_n \) must also be finite hence converge. Equivalently, if \( \sum_{n=1}^{\infty} a_n \) diverges, that is is infinite, then \( \sum_{n=1}^{\infty} b_n \) must also be infinite that is diverge. These are the only two situations under which we can conclude.

Remark 8.2.14 In the above theorem, the statement \( a_n \leq b_n \) \( \forall n \in \mathbb{N} \) can be replaced by \( a_n \leq b_n \) for every \( n \) from some point on. In other words, it is alright if the inequality is not satisfied by the first few terms of the series.
Remark 8.2.15 It is easier to remember the theorem in this form: Given two series of positive terms. If the series of larger terms converges, so does the series of smaller terms. Equivalently, if the series of smaller terms diverges, so does the series of larger terms.

The idea here is that if we suspect that the given series converges, we need to find a series with larger terms which we know converges. Conversely, if we suspect the given series diverges, we need to find a series with smaller terms which we know diverges.

It is also important to note that the inequality does not have to hold for every $n$. It just has to hold from some point on.

Example 8.2.16 Test $\sum \frac{1}{n^2 + 5}$ for convergence.

For large $n$, $n^2 + 5$ behaves like $n^2$. So, we compare $\sum \frac{1}{n^2 + 5}$ to $\sum \frac{1}{n^2}$ which is a p-series with $p = 2$, thus it converges. At this point, we cannot conclude yet that the given series converges. For this, we need to prove that the terms of $\sum \frac{1}{n^2 + 5}$ are smaller than the terms of $\sum \frac{1}{n^2}$. This is easy to see because $n^2 + 5 \geq n^2$ and therefore $\frac{1}{n^2 + 5} \leq \frac{1}{n^2}$. By the standard comparison test, it follows that $\sum \frac{1}{n^2 + 5}$ converges.

Example 8.2.17 Test $\sum \frac{\ln n}{n}$ for convergence.

Since $\ln n \geq 1$ for $n \geq 3$, it follows that $\frac{\ln n}{n} \geq \frac{1}{n}$. We know that $\sum \frac{1}{n}$ diverges.

By the standard comparison test, $\sum \frac{\ln n}{n}$ diverges.

Example 8.2.18 Test $\sum \frac{1}{2^n + 1}$ for convergence.

For large $n$, $2^n + 1$ behaves like $2^n$. We also know that $\sum \frac{1}{2^n}$ converges (geometric series with $r = \frac{1}{2}$). Thus we suspect our series converges. Since $\frac{1}{2^n + 1} \leq \frac{1}{2^n}$, we can use the standard comparison test to conclude that it does.

Example 8.2.19 Test $\sum \frac{1}{n^2 - 5}$ for convergence.

Since $n^2 - 5$ behaves like $n^2$, we suspect our series converges since $\sum \frac{1}{n^2}$ converges. However, $\frac{1}{n^2 - 5} \geq \frac{1}{n^2}$. So, we cannot use the standard comparison test. The next comparison test addresses this issue.

Theorem 8.2.20 (Limit comparison test) Suppose that $\sum a_n$ and $\sum b_n$ are two series with positive terms such that $L = \lim_{n \to \infty} \frac{a_n}{b_n}$ exists or is $\infty$ then:
8.2. TESTING SERIES WITH POSITIVE TERMS

1. If $0 < L < \infty$, the two series behave alike.

2. If $L = 0$ then convergence of $\sum b_n$ implies convergence of $\sum a_n$. Divergence of $\sum a_n$ implies divergence of $\sum b_n$.

3. If $L = \infty$ then convergence of $\sum a_n$ implies convergence of $\sum b_n$. Divergence of $\sum b_n$ implies divergence of $\sum a_n$.

**Proof.** The proof is similar to that of theorem 7.7.12 and is left as an exercise.

Intuitively, the theorem should make sense. If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ then this indicates that $b_n$ is much larger than $a_n$. Thus, $\sum b_n$ is the series with larger terms. We can use remark 8.2.15 to help us remember what the theorem allows us to conclude. Similarly, if $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$, then $a_n$ is much larger than $b_n$. So, this time $\sum a_n$ is the series with larger terms.

**Example 8.2.21** Test $\sum \frac{1}{n^2 - 5}$ for convergence.

This is the series we could not handle with the standard comparison test. Again, we compare it to $\sum \frac{1}{n^2}$. This time, we look at

$$
\lim_{n \to \infty} \frac{n^2 - 5}{n^2} = \lim_{n \to \infty} \frac{n^2}{n^2 - 5}
$$

$$
= 1 \text{ by l'Hôpital’s rule.}
$$

Since $\sum \frac{1}{n^2}$ we conclude by the limit comparison test that $\sum \frac{1}{n^2 - 5}$ also converges.

**8.2.3 Exercises**

1. Suppose that $\Sigma a_n$ and $\Sigma b_n$ are series with positive terms and $\Sigma b_n$ is known to be convergent.

   (a) If $a_n > b_n$ for all $n$, what can be said about $\Sigma a_n$? Why?

   (b) If $a_n < b_n$ for all $n$, what can be said about $\Sigma a_n$? Why?

2. It is important to distinguish between $\sum_{n=1}^{\infty} n^b$ and $\sum_{n=1}^{\infty} b^n$. What name is given to the first series? To the second? For what value of $b$ do the first series converge? For what value of $b$ do the second series converge?

3. Use the integral test to determine if the integrals below converge or diverge.
CHAPTER 8. INFINITE SERIES

(a) \( \sum_{n=1}^{\infty} \frac{1}{n^4} \)

(b) \( \sum_{n=1}^{\infty} ne^{-n} \)

4. Use a comparison test to determine if the series below converge or diverge.

(a) \( \sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1} \)

(b) \( \sum_{n=1}^{\infty} \frac{\cos^2 n}{n^3} \)

(c) \( \sum_{n=1}^{\infty} \frac{n}{n^3 - 1} \)

(d) \( \sum_{n=1}^{\infty} \frac{1 + \cos n}{n^2} \)

5. Determine whether the series below converge or diverge using any of the tests studied so far.

(a) \( 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \ldots \)

(b) \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \)

(c) \( \sum_{n=1}^{\infty} \frac{5}{2 + 3^n} \)

(d) \( \sum_{n=1}^{\infty} \frac{n + 1}{n^2} \)

(e) \( \sum_{n=1}^{\infty} \frac{n^2 + 1}{n^4 + 1} \)

6. Find the values of \( p \) for which \( \sum_{n=2}^{\infty} \frac{1}{n (\ln n)^p} \) is convergent.

7. Prove theorem 8.2.20

8. Suppose \( a_n \geq 0 \forall n \in \mathbb{N} \). Show that if \( \sum_{n=1}^{\infty} a_n \) converges then \( \sum_{n=1}^{\infty} a_n^2 \) also converges.

9. Suppose \( a_n \geq 0 \forall n \in \mathbb{N} \). Show that if \( \sum_{n=1}^{\infty} a_n \) converges then \( \sum_{n=1}^{\infty} \frac{\sqrt{an}}{n} \) also converges.
8.2. TESTING SERIES WITH POSITIVE TERMS

8.2.4 Hints for the Exercises

1. Suppose that $\Sigma a_n$ and $\Sigma b_n$ are series with positive terms and $\Sigma b_n$ is known to be convergent.

   (a) If $a_n > b_n$ for all $n$, what can be said about $\Sigma a_n$? Why?
       Hint: none

   (b) If $a_n < b_n$ for all $n$, what can be said about $\Sigma a_n$? Why?
       Hint: none

2. It is important to distinguish between $\sum_{n=1}^{\infty} n^b$ and $\sum_{n=1}^{\infty} b^n$. What name is given to the first series? To the second? For what value of $b$ do the first series converge? For what value of $b$ do the second series converge?
   Hint: none

3. Use the integral test to determine if the integrals below converge or diverge.

   (a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$
      Hint: none

   (b) $\sum_{n=1}^{\infty} ne^{-n}$

4. Use a comparison test to determine if the series below converge or diverge.

   (a) $\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1}$
      Hint: compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$

   (b) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^3}$
      Hint: recall that $0 \leq \cos^2 n \leq 1$

   (c) $\sum_{n=1}^{\infty} \frac{n}{n^3 - 1}$
      Hint: compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$

   (d) $\sum_{n=1}^{\infty} \frac{1 + \cos n}{n^2}$
      Hint: recall that $0 \leq 1 + \cos n \leq 2$

5. Determine whether the series below converge or diverge using any of the tests studied so far.
298  CHAPTER 8. INFINITE SERIES

(a)  $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \ldots$
Hint: think p-series

(b)  $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$
Use the integral test.

(c)  $\sum_{n=1}^{\infty} \frac{5}{2 + 3^n}$
Hint: Compare to $\sum_{n=1}^{\infty} \frac{5}{3^n}$

(d)  $\sum_{n=1}^{\infty} \frac{n + 1}{n^2}$
Hint: compare to $\sum_{n=1}^{\infty} \frac{1}{n}$

(e)  $\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^4 + 1}$
Hint: compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$

6. Find the values of $p$ for which $\sum_{n=2}^{\infty} \frac{1}{n \ln n^p}$ is convergent.
Hint: use the integral test.

7. Prove theorem 8.2.20
Hint: Look at the proof of theorem 7.7.12

8. Suppose $a_n \geq 0 \forall n \in \mathbb{N}$. Show that if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n^2$ also converges.
Hint: If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n \to \infty} a_n = 0$ which means that from some point on $a_n < 1$ thus $a_n^2 < a_n$.

9. Suppose $a_n \geq 0 \forall n \in \mathbb{N}$. Show that if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ also converges.
Hint: expand $\left(\sqrt{a_n} - \frac{1}{n}\right)^2$ and use theorems in these notes.
Bibliography


