Abstract

This document is a continuation of the previous document on ray tracing. It studies the intersection of a ray with triangles, boxes, spheres and cylinders. The images which appear in this document are taken from [SP1].

1 Ray Tracing

In this document we will look at the intersection of a ray with specific objects. The previous handout discussed how to find the equation of a ray. In this document, we will assume we have a ray. We are simply interested in knowing if the ray intersects with a given object. If it does, we want to know at which point.

Let us assume our ray is given by the parametric equation

\[ p(t) = q + tv \]

where \( q \) is the initial point of the ray (it will be the camera position most of the time), and \( v \) is the direction of the ray. Finding the intersection of the ray with an object is finding the value of \( t \) at which the ray intersects with the object. Recall that the smaller \( t \) is, the closer to \( q \) the point of intersection is.

1.1 Intersection of a ray with a triangle

This problem is solved in two steps. First, we find the intersection of the ray with the plane containing the triangle. Then, we determine if the point of intersection is inside the triangle. The latter is done using barycentric coordinates. See the handout on barycentric coordinates to review their properties, and how to compute them.
1.1.1 Quick Review: 3D barycentric coordinates

You will recall that given a triangle \( T = (a, b, c) \) in counterclockwise order and a point \( p \), the barycentric coordinates of \( p \) with respect to the vertices of the triangle \( T \) are the numbers \( \alpha, \beta, \gamma \) such that

\[
p = \alpha a + \beta b + \gamma c
\]

subject to the constraint

\[
\alpha + \beta + \gamma = 1
\]

The Barycentric coordinate are defined for all points in the plane. They have several nice features:

1. A point \( p \) is inside the triangle defined by \( a, b, c \) if and only if
   \[
   0 < \alpha < 1 \\
   0 < \beta < 1 \\
   0 < \gamma < 1
   \]
   This is very important. It provides an easy way to test if a point is inside a triangle.

2. If one of the barycentric coordinates is 0 and the other two are between 0 and 1, the corresponding point \( p \) is on one of the edges of the triangle.

3. If two of the barycentric coordinates are zero and the third is 1, the point \( p \) is at one of the vertices of the triangle.

4. By changing the values of \( \alpha, \beta, \gamma \) between 0 and 1, the point \( p \) will move smoothly inside the triangle.

1.1.2 Intersection between a ray and the plane containing a triangle

Let \( T = (a, b, c) \) be a triangle and assume the vertices are in counterclockwise order. We want to find if the ray \( p(t) = q + tv \) intersects with the plane containing \( T \).

You will recall that the equation of the plane with normal \( n \), containing a point \( a \) is given by \( n \cdot p - n.a = 0 \). In our case, we have \( n = (b - a) \times (c - a) \). Combining the two equal gives us

\[
\begin{align*}
n.(q + tv) & = n.a \\
n.q + tn.v & = n.a \\
t & = \frac{n.a - n.q}{n.v}
\end{align*}
\]

Recall that \( t > 0 \). Also, if \( n.v = 0 \), the ray is parallel to the plane containing the triangle, thus they will never intersect. When implementing the algorithm, \( n.v \) should be computed first. If it is found to be different from 0, then \( t \) can be computed.
1.1.3 Determining if the point is inside the triangle

Once $t$ and thus $p$ are found, to determine if $p$ is inside the triangle, we proceed as follows:

1. Find $\alpha, \beta, \gamma$ the barycentric coordinates of $p$ with respect to the vertices of $T$.
2. The values of $\alpha, \beta, \gamma$ will determine where $p$ is with respect to $T$ as explained above and in the handout on barycentric coordinates.

1.2 Intersection of a ray with a Sphere

Again, our ray is given by $p(t) = q + tv$, we will use the following notation:

\[ p = (p_x, p_y, p_z) \]
\[ q = (q_x, q_y, q_z) \]
\[ v = (v_x, v_y, v_z) \]

Recall that the equation of the sphere of radius $r$ centered at the point $C = (l, m, n)$ is given by

\[ (x - l)^2 + (y - m)^2 + (z - n)^2 = r^2 \]

A point is on both the sphere and the ray if the following equation is satisfied

\[ (p_x - l)^2 + (p_y - m)^2 + (p_z - n)^2 = r^2 \]

That is

\[ (q_x + tv_x - l)^2 + (q_y + tv_y - m)^2 + (q_z + tv_z - n)^2 = r^2 \]

Expanding the squares and collecting gives

\[ at^2 + bt + c = 0 \]

where

\[ a = v.v \]
\[ b = 2v \cdot (q - C) \]
\[ c = 2(q - C)^2 - r^2 \]

**Remark 1** Remember that if $v$ is a vector, then $v^2$ means $v.v$.

Let $\Delta = b^2 - 4ac$ be the discriminant of equation 3. We have the following:

1. $\Delta > 0$, equation 3 has two real solutions $t = \frac{-b + \sqrt{\Delta}}{2a}$. The ray intersects the sphere in two points. We will only see the one for which $t$ is the smallest.
2. $\Delta = 0$, the ray touches the sphere at one point given by $t = -\frac{b}{2a}$.

3. $\Delta < 0$, the ray does not intersect the sphere.

If the coordinates of the point of intersection are $(x_p, y_p, z_p)$, the unit normal at the point will be the vector of coordinates 

$$
\left( \frac{x_p - l}{r}, \frac{y_p - m}{r}, \frac{z_p - n}{r} \right)
$$

(5)

2 Assignment

1. Using the techniques discussed in this document, describe how you would find the intersection of a ray with a box.

2. Verify equations 3 and 4.

3. Prove equation 5.

3 Resources

This is a list of books and other resources I used to compile these notes.

References


