The Mean Value Theorem and Antiderivatives

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1 The Mean Value Theorem

The Mean Value Theorem (MVT) is the most important theorem about derivatives. It is needed to prove some of the most commonly used facts about differentiable functions. For example, it is needed to prove that if a function, \( f \), is differentiable on the interval \((a, b)\) and \( f'(x) \geq 0 \) for all \( x \in (a, b) \), then \( f \) is increasing on \((a, b)\).

Before proving the MVT (Theorem 3), we will first prove Rolle’s Theorem (which will be used in proving the MVT).

**Theorem 1 (Rolle’s Theorem)** Suppose the function \( f \) is differentiable at each point of the interval \((a, b)\) and continuous at each point of the interval \([a, b]\). Suppose also that \( f(a) = f(b) \). Then there exists a point \( c \in (a, b) \) such that \( f'(c) = 0 \).

**Proof.** First, we consider the possibility that \( f \) is constant on \([a, b]\). In this case, we have \( f'(x) = 0 \) for all \( x \in (a, b) \) and, hence, the conclusion of the theorem is obviously true in this case.

Next, suppose that \( f \) is not constant on \([a, b]\). Then, either there exists a point \( d \in (a, b) \) such that \( f(d) > f(a) \) or there exists a point \( d \in (a, b) \) such that \( f(d) < f(a) \). Let’s suppose that there exists a point \( d \in (a, b) \) such that \( f(d) > f(a) \). (The other possibility is handled similarly.) In this case, since \( f \) is continuous on \([a, b]\), we know that there exists a point \( c \in (a, b) \) such that \( f(c) \geq f(x) \) for all \( x \in [a, b] \). Since \( c \in (a, b) \), we know that \( f \) is
differentiable at $c$. Also,

$$\frac{f(x) - f(c)}{x - c} \leq 0 \text{ for all } x > c$$

and

$$\frac{f(x) - f(c)}{x - c} \geq 0 \text{ for all } x < c$$

which means that

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0.$$

Since $0 \leq f'(c) \leq 0$, we conclude that $f'(c) = 0$. $\blacksquare$

**Exercise 2** The hypotheses of Rolle’s Theorem are that

1. $f$ is differentiable at each point of $(a, b)$.
2. $f$ is continuous at each point of $[a, b]$.
3. $f(a) = f(b)$

Give examples which show that each of these hypotheses is needed in order for Rolle’s Theorem to be true. That is, give examples of functions which satisfy only two out of the three above hypotheses and for which the conclusion of Rolle’s Theorem is not true.

**Theorem 3 (Mean Value Theorem)** Suppose that the function $f$ is differentiable at each point of the interval $(a, b)$ and continuous at each point of the interval $[a, b]$. Then there exists a point $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Proof.** Consider the function $F$ defined for each $x \in [a, b]$ by

$$F(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a).$$
Clearly, $F$ is differentiable at each point of $(a, b)$ and continuous at each point of $[a, b]$. Also, $F(a) = 0 = F(b)$. Thus, by Rolle’s Theorem, there exists a point $c \in (a, b)$ such that $F'(c) = 0$. Since

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \text{ for all } x \in (a, b),$$

we have

$$0 = F'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

from which we conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

\[\Box\]

**Remark 4** In graphical terms, the Mean Value Theorem states the existence of a point $c \in (a, b)$ where the slope of the tangent line to the graph of $f$ at the point $(c, f(c))$ is the same as the slope of the secant line joining the points $(a, f(a))$ and $(b, f(b))$. Hence, the Mean Value Theorem is a theorem relating secant lines and tangent lines.

**Example 5** Let us consider the function $f(x) = x^2 + x - 2$ on the interval $[-3, 3]$. At the endpoints of this interval, we have

$$f(-3) = (-3)^2 + (-3) - 2 = 4$$

and

$$f(3) = (3)^2 + 3 - 2 = 10.$$ 

The secant line joining the points $(-3, 4)$ and $(3, 10)$ has slope

$$\frac{10 - 4}{3 - (-3)} = \frac{6}{6} = 1.$$ 

The Mean Value Theorem tells us that there must exist a point $c \in (-3, 3)$ such that the tangent line to the graph of $f$ at the point $(c, f(c))$ also has slope 1. In fact it easy to find such a point by setting $f'(x) = 1$ and solving for $x$. Since $f'(x) = 2x + 1$, we solve

$$2x + 1 = 1$$

to obtain $x = 0$. This means that the tangent line to the graph of $f$ at the point $(0, f(0)) = (0 - 2)$ has slope 1. See Figure 1.
Exercise 6  For each of the following functions \( f \) and intervals \([a, b]\):

a. Find the slope, \( m \), of the secant line joining the points \((a, f(a))\) and \((b, f(b))\).

b. Find a point \( c \in (a, b) \) such that \( f'(c) = m \).

c. Graph \( f \) together with its tangent line at the point \((c, f(c))\). (Also, draw in the secant line joining the points \((a, f(a))\) and \((b, f(b))\).)

1. \( f(x) = x^2 + x - 2 \), \([a, b] = [0, 2] \).
2. \( f(x) = x^2 + x - 2 \), \([a, b] = [-3, 2] \).
3. \( f(x) = x^3 - 12x \), \([a, b] = [0, 2] \).
4. \( f(x) = \sqrt{x} \), \([a, b] = [0, 9] \).
5. \( f(x) = \cos x \), \([a, b] = [0, 2\pi] \).

Exercise 7  The hypotheses of the Mean Value Theorem are that

1. \( f \) is differentiable at each point of \((a, b)\).
2. \( f \) is continuous at each point of \([a, b]\).

Give examples which show that each of these hypotheses is needed in order for the Mean Value Theorem to be true. That is, give examples of functions which satisfy only one out of the two above hypotheses and for which the conclusion of the Mean Value Theorem is not true.
We now give two very important corollaries of the Mean Value Theorem. Corollary 9 will be of prime importance in our study of antiderivatives.

**Corollary 8** Suppose that the function $H$ is differentiable at each point of $(a,b)$ and suppose that $H'(x) = 0$ for all $x \in (a,b)$. Then $H$ is constant on $(a,b)$ (meaning that there is a constant, $C$, such that $H(x) = C$ for all $x \in (a,b)$).

**Proof.** Let $x_0$ be a point in $(a,b)$ and let $x$ be a point in $(a,b)$ with $x_0 < x$. Since $H$ is differentiable at each point of $(x_0, x)$ and continuous at each point of $[x_0, x]$, then by the MVT there exists a point $c \in (x_0, x)$ such that

$$H'(c) = \frac{H(x) - H(x_0)}{x - x_0}.$$ 

Since $H'(c) = 0$, then $H(x) = H(x_0)$. Since $x$ was chosen arbitrarily from $(x_0, b)$, this shows that $H(x) = H(x_0)$ for all $x \in (x_0, b)$. By similar reasoning, we can show that $H(x) = H(x_0)$ for all $x \in (a, x_0)$. We conclude that $H(x) = H(x_0)$ for all $x \in (a,b)$, i.e., $H$ is constant on $(a,b)$. 

**Corollary 9** Suppose that $F$ and $G$ are functions that are both differentiable at each point of $(a,b)$ and suppose that $F'(x) = G'(x)$ for all $x \in (a,b)$. Then there exists a constant, $C$, such that $G(x) = F(x) + C$ for all $x \in (a,b)$.

**Proof.** The function $H$ defined by $H(x) = G(x) - F(x)$ is differentiable at each point of $(a,b)$ and $H'(x) = G'(x) - F'(x) = 0$ at each point of $(a,b)$. Thus, by Corollary 8, $H$ is constant on $(a,b)$. This means that there exists a constant, $C$, such that $H(x) = C$ for all $x \in (a,b)$. Since $H(x) = G(x) - F(x)$ for all $x \in (a,b)$, we have $G(x) - F(x) = C$ for all $x \in (a,b)$ which means that $G(x) = F(x) + C$ for all $x \in (a,b)$.

**Remark 10** Corollary 9 tells us that if two functions have the same derivative on an interval $(a,b)$, then the two functions differ by a constant. In other words, the graphs of the two functions are vertical translations of each other. This idea is illustrated in the two examples that follow.

**Example 11** Consider the functions $F(x) = -4x + 14$ and $G(x) = -4x + 6$. These functions have the same derivative: $F'(x) = -4$ and $G'(x) = -4$. Also, we observe $G(x) = F(x) - 8$; i.e., the graph of $G$ is the graph of $F$ translated downward by 8 units. See Figure 2.
Figure 2: $G(x) = -4x + 6$ and $F(x) = -4x + 14$

**Example 12** The functions $F(x) = \sin x$ and $G(x) = \sin x + 2$ have the same derivative: $F'(x) = \cos x$ and $G'(x) = \cos x$. Also, we observe that $G(x) = F(x) + 2$ which means that the graphs of $F$ and $G$ are vertical translations of each other (differing by 2 units). See Figure 3.

Figure 3: $F(x) = \sin x$ and $G(x) = \sin x + 2$

Another familiar corollary of the Mean Value Theorem that we have used frequently is:

**Corollary 13** Suppose that the function $f$ is differentiable at each point of the interval $(a,b)$ and suppose that $f'(x) \geq 0$ for all $x \in (a,b)$. Then $f$ is increasing on $(a,b)$.
**Proof.** We must show that if \( x_1 \) and \( x_2 \) are any two points in \((a, b)\) with \( x_1 < x_2 \), then \( f(x_1) \leq f(x_2) \). To this end, let \( x_1 \) and \( x_2 \) be two points in \((a, b)\) with \( x_1 < x_2 \). Since \( f \) is differentiable on \((x_1, x_2)\) and continuous on \([x_1, x_2]\), then by the MVT there exists a point \( c \in (x_1, x_2) \) such that

\[
f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.
\]

Since \( f'(c) \geq 0 \) and \( x_2 - x_1 > 0 \), it must also be true that \( f(x_2) - f(x_1) \geq 0 \), i.e., \( f(x_1) \leq f(x_2) \). □

**Exercise 14** Suppose that the function \( f \) is differentiable at each point of the interval \((a, b)\) and suppose that \( f'(x) \leq 0 \) for all \( x \in (a, b) \). Prove that \( f \) is decreasing on \((a, b)\).

## 2 Antiderivatives

As the name suggests, anti–differentiation is an “undoing” of the differentiation process. For example, the derivative of the function \( F(x) = 3x^2 - 5x + 4 \) is the function \( f(x) = 6x - 5 \), whereas an antiderivative of the function \( f(x) = 6x - 5 \) is the function \( F(x) = 3x^2 - 5x + 4 \).

**Definition 15** If the function \( F \) is differentiable at each point of the interval \((a, b)\) and \( F'(x) = f(x) \) for all \( x \in (a, b) \), then \( f \) is called the **derivative** of \( F \), and \( F \) is called an **antiderivative** of \( f \).

Something that should be pointed out immediately is the choice of articles that are used when speaking about derivatives and antiderivatives. We say the **derivative** (using the definite article “the”) when speaking about derivatives, but say an **antiderivative** (using the indefinite article “an”) when speaking about antiderivatives. Why is this so? The reason is the definite article “the” is used to refer to something that is unique (meaning the only one of its kind). For example, we say the President of the United States and the Empire State Building because there is only one Empire State Building and only one President of the United States (at any given time). The indefinite articles “a” and “an” are used to refer to something that is not unique. For example, we say “a parrot” when we want to make a general statement about parrots such as “A parrot makes a nice pet”.

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Keeping the above discussion of grammar in mind, why do we say that the function \( f(x) = 6x - 5 \) is the derivative of the function \( F(x) = 3x^2 - 5x - 4 \), but say that the function \( F(x) = 3x^2 - 5x + 4 \) is an antiderivative of the function \( f(x) = 6x - 5 \)? The reason is that \( F(x) = 3x^2 - 5x - 4 \) has a unique derivative. It is the function \( f(x) = 6x - 5 \) and no other! However the function \( f(x) = 6x - 5 \) does not have a unique antiderivative. One antiderivative of the function \( f(x) = 6x - 5 \) is the function \( F(x) = 3x^2 - 5x - 4 \), but the function \( G(x) = 3x^2 - 5x + 46 \) is also an antiderivative of \( f \) (as the reader can easily check, because \( G'(x) = f(x) \)). In fact, if \( C \) is any constant whatsoever, then the function \( F(x) = 3x^2 - 5x + C \) is an antiderivative of \( f \). Thus, \( f \) has infinitely many antiderivatives.

### 2.1 Families of Antiderivatives

As is evident from the preceding discussion, antiderivatives are not unique. For this reason, we use the term *family of antiderivatives* when referring to the set of all functions that are antiderivatives of a given function \( f \). Thus, we would say for example that

the family of antiderivatives of the function \( f(x) = 6x - 5 \) is

the set of all functions of the form \( F(x) = 3x^2 - 5x + C \) (where \( C \) can be any constant).

Often, we drop the “where \( C \) is any constant” statement at the end because that is taken to be understood.

Before proceeding, an important discussion is in order: It is clear that all functions of the form

\[
F(x) = 3x^2 - 5x + C
\]

(1)

are antiderivatives of \( f(x) = 6x - 5 \), but how can we be sure that these functions are the only antiderivatives of \( f \)? In other words, might there be a function, \( G \), that looks nothing like the function (1) but such that \( G \) is an antiderivative of \( f \)? The answer is “No” and the reason lies in Corollary 9 of the Mean Value Theorem. We state this result in general in the following theorem.

**Theorem 16** If the function \( F \) is an antiderivative of the function \( f \) on the interval \((a, b)\), then **every** antiderivative of \( f \) on \((a, b)\) is a function of the form \( G(x) = F(x) + C \) (where \( C \) is a constant).
The proof of Theorem 16 follows directly from Corollary 9: If $F$ is an antiderivative of $f$ on $(a, b)$, then $F'(x) = f(x)$ for all $x \in (a, b)$. However, if $G$ is an antiderivative of $f$ on $(a, b)$, then $G'(x) = f(x)$ for all $x \in (a, b)$. This means that $F'(x) = G'(x)$ for all $x \in (a, b)$ and hence, by Corollary 9, there is a constant $C$ such that $G(x) = F(x) + C$ for all $x \in (a, b)$.

Thanks to Theorem 16, the problem of finding all antiderivatives of a function $f$ reduces to the problem of finding just a single antiderivative, $F$, of $f$. If $F$ is an antiderivative of $f$, then any antiderivative, $G$, of $f$ has the form $G(x) = F(x) + C$ where $C$ is a constant.

**Example 17 Problem:** Find the family of antiderivatives of the function $f(x) = 2x$.

**Solution:** After a moment of thought, we realize that the function $F(x) = x^2$ is an antiderivative of $f$. This means (by Theorem 16) that every antiderivative, $G$, of $f$ has the form $G(x) = x^2 + C$. We conclude that the family of antiderivatives of the function $f(x) = 2x$ is the set of all functions of the form $F(x) = x^2 + C$.

**Example 18 Problem:** Find the family of antiderivatives of the function $f(x) = \cos x$.

**Solution:** Calling on our knowledge of the derivatives of trigonometric functions, we remember that the function $F(x) = \sin x$ is an antiderivative of $f$. Thus, the family of antiderivatives of the function $f(x) = \cos x$ is the set of all functions of the form $F(x) = \sin x + C$.

### 2.2 Antiderivatives of Commonly-Used Functions

Each “differentiation fact” that we know gives us a gives us a corresponding “antidifferentiation fact”. For example, the fact that the derivative of $F(x) = x^2$ is $f(x) = 2x$ gives us the fact that an antiderivative of $f(x) = 2x$ is $F(x) = x^2$ and, more generally, gives us the fact that the family of antiderivatives of $f(x) = 2x$ is the set of all functions of the form $F(x) = x^2 + C$.

Below, we summarize some of the main differentiation facts that we have learned about commonly-used functions and we give the corresponding antidifferentiation facts.

**Differentiation Fact:** If $n$ is a constant, then the derivative of $F(x) = x^n$ is $f(x) = nx^{n-1}$.
**Corresponding Antidifferentiation Fact:** If \( n \) is a constant, then the family of antiderivatives of \( f(x) = nx^{n-1} \) consists of all functions of the form \( F(x) = x^n + C \).

**Differentiation Fact:** The derivative of \( F(x) = \sin x \) is \( f(x) = \cos x \).

**Corresponding Antidifferentiation Fact:** The family of antiderivatives of \( f(x) = \cos x \) consists of all functions of the form \( F(x) = \sin x + C \).

**Differentiation Fact:** The derivative of \( F(x) = \cos x \) is \( f(x) = -\sin x \).

**Corresponding Antidifferentiation Fact:** The family of antiderivatives of \( f(x) = -\sin x \) consists of all functions of the form \( F(x) = \cos x + C \).

**Differentiation Fact:** The derivative of \( F(x) = e^x \) is \( f(x) = e^x \).

**Corresponding Antidifferentiation Fact:** The family of antiderivatives of \( f(x) = e^x \) consists of all functions of the form \( F(x) = e^x + C \).

Of course, less obvious differentiation facts that arise from differentiation “rules” such as the Constant Multiple Rule, Product Rule, Chain Rule, etc., also give rise to corresponding antidifferentiation facts and these are sometimes trickier to spot if we are not given the differentiation fact first! A few of these “trickier” examples are given below. Although these examples usually at first seem difficult to most calculus students, rest assured that they become easier with practice.

**Example 19 Problem:** Determine the family of antiderivatives of the function \( f(x) = x^3 \).

**Solution:** A first guess might be that \( F(x) = x^4 \) is an antiderivative of \( f(x) = x^3 \), but this is not right because the derivative of \( x^4 \) is \( 4x^3 \) (not \( x^3 \)). In order to not have the extra factor of 4, we need to multiply \( 4x^3 \) by \( 1/4 \). Using the Constant Multiple Rule, we discover that in fact \( F(x) = \frac{1}{4}x^4 \) is an antiderivative of \( f(x) = x^3 \) and we conclude that the family of antiderivatives of \( f(x) = x^3 \) consists of all functions of the form \( F(x) = \frac{1}{4}x^4 + C \).

**Example 20 Problem:** Determine the family of antiderivatives of the function \( f(x) = -8x^4 - 2x^3 + x^2 - 6x - 6 \).

**Solution:** After carefully studying the previous example (and perhaps trying a few more similar examples), we realize that if \( n \) is a constant (with
n \neq -1), then an antiderivative of \( x^n \) is \( \frac{1}{n+1}x^{n+1} \). Thus, for example, an antiderivative of \( x^4 \) is \( \frac{1}{5}x^5 \). Using the Constant Multiple Rule, we then see that an antiderivative of \(-8x^4\) is \(-8 \cdot \frac{1}{5}x^5\) which is the same as \(-\frac{8}{5}x^5\). Treating each term in \( f \) in a similar way and using the Sum Rule, we conclude that the family of antiderivatives of \( f (x) = -8x^4 - 2x^3 + x^2 - 6x - 6 \) consists of all functions of the form

\[ F(x) = -\frac{8}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 - 3x^2 - 6x + C. \]

It is easy to check that our answer is correct; i.e. to check that \( F'(x) = f(x) \).

**Example 21** **Problem:** Determine the family of antiderivatives of the function \( f(x) = 12\sin x \).

**Solution:** One of our “basic” antidifferentiation facts is that an antiderivative of \(-\sin x\) is \( \cos x \). By keeping this fact in mind and using the Constant Multiple Rule, we discover that an antiderivative of \( \sin x \) is \(-\cos x \). Using the Constant Multiple Rule again, we see that an antiderivative of \( 12\sin x \) is \(-12\cos x \). Thus, the family of antiderivatives of \( f(x) = 12\sin x \) consists of all functions of the form \( F(x) = -12\cos x \).

**Example 22** **Problem:** Determine the family of antiderivatives of the function \( f(x) = \sec^2 x \).

**Solution:** If you remember the “somewhat lesser used” differentiation facts about trigonometric functions, the answer to this problem should jump out at you. The fact that

\[ \frac{d}{dx} (\tan x) = \sec^2 x \]

allows us to conclude that the family of antiderivatives of \( f(x) = \sec^2 x \) consists of all functions of the form \( F(x) = \tan x + C \).

**Example 23** **Problem:** Determine the family of antiderivatives of the function \( f(x) = e^{4x} \).

**Solution:** To do this problem, we recall the fact that if \( k \) is a constant, then the derivative of \( e^{kx} \) is \( ke^{kx} \). A first guess at an antiderivative of \( e^{4x} \) might be \( e^{4x} \) – but this is wrong because the derivative of \( e^{4x} \) is \( 4e^{4x} \). We once again “fix things up” by using the Constant Multiple Rule. The derivative of \( \frac{1}{4}e^{4x} \) is \( \frac{1}{4} \cdot 4e^{4x} = e^{4x} \). Thus the family of antiderivatives of \( f(x) = e^{4x} \) consists of all functions of the form \( F(x) = \frac{1}{4}e^{4x} + C \).
Example 24 (A Hard Problem) The final example that we give might be classified as “unreasonably hard”. We give it, not to scare the reader, but to illustrate that the antidifferentiation process is indeed usually much harder than the differentiation process.

Suppose that we are asked to find the family of antiderivatives of the function

\[ f(x) = \frac{2x \cos x + \sin x}{2\sqrt{x}}. \]

As promised, this is a hard problem. Give it a try before going on to the next example.

Example 25 (An Easy Problem) In contrast to the last example, we now look at an easy problem. It is not an antidifferentiation problem – but a differentiation problem.

**Problem:** Find the derivative of the function \( F(x) = \sqrt{x} \sin x \).

**Solution:** Using the fact that \( F(x) = f_1(x) \cdot f_2(x) \) (a product) where \( f_1(x) = \sqrt{x} \) and \( f_2(x) = \sin x \), we use the Product Rule to obtain

\[
F'(x) = f_1(x) \cdot f_2'(x) + f_1'(x) \cdot f_2(x) \\
= \sqrt{x} \cdot \cos x + \frac{1}{2\sqrt{x}} \cdot \sin x.
\]

Our differentiation problem has been solved (rather routinely) by using the Product Rule. Looking at the result, the reader might suspect that there is a connection between this problem and the “hard problem” of the previous example—and indeed there is. By doing some algebra, we can write

\[
F'(x) = \sqrt{x} \cdot \cos x + \frac{1}{2\sqrt{x}} \cdot \sin x \\
= \frac{2\sqrt{x} \cdot \sqrt{x} \cdot \cos x}{2\sqrt{x}} + \frac{1}{2\sqrt{x}} \cdot \sin x \\
= \frac{2x \cos x}{2\sqrt{x}} + \frac{1}{2\sqrt{x}} \cdot \sin x \\
= \frac{2x \cos x + \sin x}{2\sqrt{x}}.
\]

thus observing that the \( F'(x) = f(x) \) where \( f \) is the function given in the previous example. If we had known this, then the previous example would not
have been hard! The family of antiderivatives of the function

\[ f(x) = \frac{2x \cos x + \sin x}{2\sqrt{x}} \]

consists of all functions of the form

\[ F(x) = \sqrt{x} \sin x + C. \]

**Exercise 26** Find the family of antiderivatives of each of the following functions, \( f \). State your results using complete sentences of the form “The family of antiderivatives of the function \( f(x) = \ldots \) consists of all functions of the form \( F(x) = \ldots \).”

1. \( f(x) = 9 \)
2. \( f(x) = 0 \)
3. \( f(x) = x^6 \)
4. \( f(x) = 10x^9 \)
5. \( f(x) = -3x^8 + 4x^6 - x^5 + x + 1 \)
6. \( f(x) = -\cos x \)
7. \( f(x) = -\frac{1}{2} \sin x \)
8. \( f(x) = 3e^x \)
9. \( f(x) = 3e^x + 4x^2 - 12 \)
10. \( f(x) = e^{x-4} \)
11. \( f(x) = \csc^2 x \)
12. \( f(x) = \sin x - \cos x \)
13. \( f(x) = e^{6x} \)
14. \( f(x) = 3e^{6x} + 2x + 1 \)
15. \( f(x) = e^x + e^{-x} \)
16. \( f(x) = e^x - e^{-x} \)

17. \( f(x) = 3(x^3 - 5x^2 + 3)^2(3x^2 - 10x) \)

18. \( f(x) = 2 \sin x \cos x \)

19. \( f(x) = e^x (\sin x + \cos x) \)

20. \( f(x) = \cos x - x \sin x \)

Exercise 27 If you found numbers 19 and 20 of the previous exercise set to be difficult, try doing these two problems first.

1. Find the derivative of the function \( F(x) = e^x \sin x \).

2. Find the derivative of the function \( F(x) = x \cos x \).